Quasiisometric Rigidity in Rank-One Symmetric Spaces à la Pansu

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Abstract. This thesis investigates a rigidity property of quasiisometries of quaternionic hyperbolic spaces and the octonionic hyperbolic plane. Its goal is to prove that every quasiisometry of these spaces differs from an isometry only by a map that move points by a globally bounded amount. Pansu gave a proof of this result that relies on an investigation of the extensions of quasiisometries to the boundaries at infinity of the rank-one symmetric spaces. The boundaries at infinity naturally have a Carnot group structure, and it can be shown that quasiisometries extend to quasiconformal homeomorphisms. Pansu introduced a framework in which differentials of quasiconformal homeomorphisms between Carnot groups can be defined, and it is in these differentials that the rigidity becomes evident. In this thesis we provide an elaboration of Pansu's proof together with various explicit presentations of the structures encountered along the way. In addition, we present a geometric interpretation of a part of Pansu's proof, which contributes to the understanding of why the main result of this thesis does not generalise to the real and complex hyperbolic spaces.

Contents

1 Introduction **1**

2 Carnot groups **4**

2.1 Definition and some useful properties . . . 4
2.2 The subRiemannian structure of Carnot groups . . . 5
2.3 Differentiability on Carnot groups . . . 7

3 Differentiability of quasiconformal homeomorphisms **8**

3.1 Outline of the proof . . . 8
3.2 Reduction to the case of curves . . . 9
3.3 Differentiability of rectifiable curves . . . 15
3.4 Differentiability of quasiconformal homeomorphisms . . . 18

4 Absolute continuity of quasiconformal homeomorphisms **23**

4.1 Bijectivity of the differential . . . 23
4.2 Absolute continuity on lines . . . 28

5 Quasiisometries of rank-one symmetric spaces **34**

5.1 The rank-one symmetric spaces . . . 34
5.2 Boundaries at infinity and their Carnot group structure . . . 36
5.3 Quasiisometries and their extensions to the boundaries . . . 39

6 Graded automorphisms of the boundaries at infinity **44**

6.1 The cases of the quaternionic hyperbolic spaces and the octonionic hyperbolic plane 44
6.2 The real and the complex case . . . 49
6.3 A geometric perspective . . . 50

7 Realising 1-quasiconformal homeomorphisms as extensions of isometries **54**

7.1 Outline of the proof . . . 54
7.2 All 1-quasiconformal homeomorphisms come from isometries . . . 55

8 Proof of the main theorem **60**

References **61**

1 Introduction

This thesis explores a rigidity property of quasiisometries of quaternionic hyperbolic spaces and the octonionic hyperbolic plane. Its goal is to prove that every quasiisometry of these spaces differs from an isometry only by a function that moves points by a bounded amount. This is a property not shared by the real and complex hyperbolic spaces. The main result is the following theorem, which is our Theorem 8.1. It has first been stated and proven by Pansu.

Theorem. [Pan89b, Theorem 1] *Every quasiisometry of a quaternionic hyperbolic space* $\mathbb{H}\mathbf{H}^n$*, where* $n \geq 2$*, respectively of the octonionic hyperbolic plane* $\mathbb{O}\mathbf{H}^2$*, lies a bounded distance away from an isometry, that is, it differs from an isometry by an application which moves points a bounded distance away.*

There are three families of hyperbolic spaces, the real hyperbolic spaces, denoted $\mathbb{R}\mathbf{H}^n$, the complex hyperbolic spaces $\mathbb{C}\mathbf{H}^n$ and the quaternionic hyperbolic spaces $\mathbb{H}\mathbf{H}^n$, with $n \geq 2$ indicating the dimension, and there is one exceptional case which is the octonionic hyperbolic plane $\mathbb{O}\mathbf{H}^2$. We will see that Pansu's theorem does not extend to the real and complex hyperbolic spaces.

A quasiisometry is a map $f\colon X \to X'$ between metric spaces with two constants L and C such that for all $x_1, x_2 \in X$ we have

$$-C + \frac{1}{L} d(x_1, x_2) \leq d'(f(x_1), f(x_2)) \leq L d(x_1, x_2) + C,$$

and $d'(y, f(X)) \leq C$ for all $y \in X'$. The study of quasiisometries is motivated by geometric group theory. Finitely generated groups equipped with word metrics naturally form metric spaces. While the specific metric structure depends on the generating set, it can be shown that changing the generating set only alters the metric space by a quasiisometry, and does not affect the geometry on larger scales. Sometimes, we can recover substantial information about the algebraic properties of a group from its quasiisometric properties. This is what we call quasiisometric rigidity.

A seminal example of this interplay is Gromov's study of group growth. His work established that groups with polynomial growth are virtually nilpotent, demonstrating how a group's growth rate can reflect its underlying geometric structure. It further implies that any group quasiisometric to a nilpotent group is itself virtually nilpotent [Gro81]. This example illustrates how large-scale geometric properties can determine algebraic characteristics across entire classes of groups.

Gromov's work has played a crucial role in highlighting the importance and richness of these concepts and demonstrating the value of their study.

Another notable result on quasiisometric rigidity is Mostow's rigidity theorem, which asserts that any isomorphism of uniform lattices in the isomorphism group $\mathrm{Isom}\left(\mathbb{R}\mathbf{H}^n\right)$ of the real hyperbolic space $\mathbb{R}\mathbf{H}^n$ for $n \geq 3$ is induced by an isometry of the corresponding hyperbolic space [Mos68; Mos73]. Uniform lattices are cocompact finitely generated subgroups of Lie groups. According to the Milnor-Švarc lemma, any lattice is quasiisometric to the Lie group itself, and an isomorphism between two lattices extends to a quasiisometry between the corresponding symmetric spaces. If this quasiisometry is within a bounded distance of an isometry, Mostow's rigidity theorem implies that the subgroups are conjugate in the case of $\mathrm{Isom}\left(\mathbb{R}\mathbf{H}^n\right)$ if $n > 2$ [Pan89b; Sch95].

Mostow's proof employs the behaviour of group actions on the boundaries at infinity, which, in the case of quasiisometries, is by quasiconformal homeomorphisms, and the characteristics of these quasiconformal homeomorphisms [Bou18]. This approach has inspired much subsequent research, such as the work of Sullivan and Tukia, which shows that any finitely generated group quasiisometric to $\mathbb{R}\mathbf{H}^n$ is isomorphic to a lattice of $\mathrm{Isom}\left(\mathbb{R}\mathbf{H}^n\right)$, up to finite index [Sul81; Tuk86]. An analogous result for the complex hyperbolic spaces has been obtained by Chow [Cho96]. The quasiconformal structure of the boundary of a hyperbolic space thus determines much of the group's algebraic structure.

This is also true for the remaining rank-one symmetric spaces, and it follows from the work of Pansu which is the topic of this thesis. Pansu's theorem establishes a quasiisometric rigidity property of uniform lattices in $\mathrm{Isom}\left(\mathbb{H}\mathbf{H}^n\right)$ or $\mathrm{Isom}\left(\mathbb{O}\mathbf{H}^2\right)$. It implies that any group quasiisometric to such a lattice is itself a uniform lattice in $\mathrm{Isom}\left(\mathbb{H}\mathbf{H}^n\right)$ or $\mathrm{Isom}\left(\mathbb{O}\mathbf{H}^2\right)$ respectively, up to taking quotients by finite normal subgroups or subgroups of finite index [Dru07].

In this thesis, we provide a proof of Pansu's theorem, largely following the original proof. It relies on the theory of differentiability of quasiconformal homeomorphisms between Carnot groups, and an investigation of the thus obtained differentials of quasiconformal homeomorphisms in particular. The connection with quasiisometries of rank-one symmetric spaces arises from the fact that we can associate with each of these spaces a boundary at infinity which can be equipped with the structure of a Carnot group. Quasiisometries can be extended to maps of the boundary, and we will see that the extensions are quasiconformal homeomorphisms. In the cases of the quaternionic hyperbolic spaces and the octonionic hyperbolic plane, there are certain constraints on the possible differentials, which will imply the main theorem.

We begin this thesis with an introduction to Carnot groups. Carnot groups are a class of Lie groups that can be equipped with Carnot-Carathéodory metrics. The structure of the resulting metric spaces allows to consider similarity transformations. In [Pan89b], Pansu introduces a notion of differentiability for maps between Carnot groups, employing these similarity transformations. This is treated in Section 2.

Among the maps between Carnot groups that are differentiable in the sense of Pansu are quasiconformal homeomorphisms. We introduce quasiconformal homeomorphisms as maps that deform spheres by bounded amounts. Maps with this property are locally well-behaved enough to be differentiable. Proving this is the goal of Section 3.

Further, we investigate the continuity properties of quasiconformal homeomorphisms. The two major observations from this section are that the differential of a quasiconformal map from a Carnot group to itself is a group automorphisms, and that quasiconformal maps are absolutely continuous along almost every line. This is our Section 4.

The reason why considering differentiability and continuity properties of quasiconformal homeomorphisms between Carnot groups is interesting in the context of this thesis is that the boundaries at infinity of the rank one symmetric spaces naturally have a Carnot group structure, and that quasiisometries extend to quasiconformal homeomorphisms on the boundary. After a brief overview of the rank-one symmetric spaces in Section 5.1, we make this precise in Section 5.2.

In Section 5.3, we show that quasiisometries map asymptotic geodesics to asymptotic quasigeodesics, which is why they can be extended to maps of the boundaries. Their quasiisometric properties cause these maps to be quasiconformal, so that they are differentiable in the sense of Pansu, and their differentials are automorphisms of the Carnot groups associated with the boundaries.

This is the point where a difference between the real and complex hyperbolic spaces and the quaternionic hyperbolic spaces and the octonionic hyperbolic plane becomes evident. In a way, there is more structure in the boundaries of the latter two, so that requiring to preserve it restricts the possible group automorphisms to similarities. This is not the case for the real and complex hyperbolic spaces. We prove this in Section 6 and give this difference a geometric interpretation. To complete the proof of the main theorem, it only remains to show that every similarity, or, in the case of $\mathbb{H}\mathbf{H}^n$ and $\mathbb{O}\mathbf{H}^2$, every differential of an extension to the boundary of a quasiisometry, is in fact the differential of an extension of an isometry of the corresponding hyperbolic space. This is done in Section 7. We further show that if the differentials coincide, then already the maps must coincide on the boundary, and since maps of hyperbolic spaces with the same extension to the boundary differ at most by bounded amounts, the main theorem follows. Finally, in Section 8, we prove our main theorem.

Our treatment follows Pansu's original proof closely, unless otherwise specified. In addition to Pansu's original work [Pan89b], we have also used an English translation by Pallier [Pal22].

2 Carnot groups

2.1 Definition and some useful properties

A Carnot group is a simply connected nilpotent Lie group whose Lie algebra admits a gradation such that the first layer generates the entire Lie algebra. It can be equipped with a so-called subRiemannian structure. Essentially, a subRiemannian structure on a manifold is a generalisation of a Riemannian structure, the main difference being that constraints are imposed on the direction of motion of curves that are used to measure distances. The distance function that is obtained from such a structure is called a Carnot-Carathéodory metric. Such a Carnot-Carathéodory metric leads to the existence of similarity transformations such as translations, homotheties and combinations thereof, which then again allow to adapt the classical notion of a differential to one that respects this structure of the Carnot group.

In the following, we will introduce Carnot groups and some of their properties. Carnot-Carathéodory metrics are discussed in Section 2.2. Pansu's notion of differentiability of maps of Carnot groups is outlined in Section 2.3. Our treatment loosely follows [LD23, Chapters 0, 6 and 8].

Definition 2.1. A *Carnot group* is a simply connected nilpotent Lie group N, together with a derivation α on the Lie algebra $\mathfrak{n}$, such that $V^1 := \ker(\alpha - 1)$ generates the Lie algebra in the sense that, if we define inductively $V^{i+1} = [V^1, V^i]$, we have

$$\mathfrak{n} = \bigoplus_{i=1}^{n} V^i.$$

Remark 2.2. A simple calculation shows that $\alpha|_{V^i} = i \cdot id_{V^i}$.

For nilpotent matrix Lie groups, the exponential map is a finite power series, and as such admits a global inverse. This allows to conclude the following.

Lemma 2.3. *[CG90, Theorem 1.2.1] Let N be a Carnot group. Then the exponential map $\exp\colon \mathfrak{n} \to N$ is a diffeomorphism.*

This further implies that the Baker-Campbell-Hausdorff series is finite and polynomial, and allows for the definition of convenient global coordinates, which we will call *exponential coordinates*, as follows. Choose a basis of the Lie algebra $X_1, \cdots, X_n$ and define

$$\begin{array}{cccc} \phi\colon & \mathbb{R}^n & \to & G \\ & (x_1, \cdots, x_n) & \mapsto & \exp\left(\sum_{i=1}^{n} x_i X_i\right). \end{array} \tag{2.1}$$

This is a global parameterisation by Lemma 2.3.

2.2 The subRiemannian structure of Carnot groups

A subRiemannian structure on a manifold can be viewed as a generalisation of a Riemannian structure, in the sense that it imposes constraints on the directions of motion for the curves that are used for defining a distance function on the manifold. We first give a formal definition and then explain how to obtain such a structure on a Carnot group.

Definition 2.4. A distribution Δ is *bracket-generating* if every tangent vector $X \in TN$ is a linear combination of $X_1, [X_2, X_3], [X_4, [X_5, X_6]], \cdots$, where $X_1, X_2, \cdots$ are tangent to Δ.

Definition 2.5. A *subRiemannian manifold* is a triple (N, Δ, g), where N is a differentiable manifold, Δ is a bracket generating distribution and g is a smooth section of positive definite quadratic forms on Δ.

Definition 2.6. If (N, Δ, g) is a subRiemannian manifold, then a curve γ in N is *horizontal* if it is piecewise smooth and $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for all t. For $p \in N$, we call $\Delta_p \subseteq T_pN$ the *horizontal subspace* at p.

Definition 2.7. For a subRiemannian manifold (N, Δ, g), we can define a distance function, the *subRiemannian distance*, by

$$d(p, q) = \inf\left\{\text{length}(\gamma)\colon \gamma \text{ horizontal from } p \text{ to } q\right\}.$$

Such a distance is also called a *Carnot-Carathéodory metric.*

Note that the condition that Δ is bracket-generating guarantees that for any pair of points p, q, there is always a horizontal curve from p to q [LD23, Theorem 3.1.17].

On a Carnot group, where we have $V^1 = \ker(\alpha - 1)$, we may set $\Delta_p = d_e L_p(V^1)$. This distribution is automatically bracket-generating, and we obtain a Carnot-Carathéodory metric by specifying an inner product g_e on V^1 and translating it to other points $p \in M$ by setting $g_p = (L_p)_* g_e$. Note that the resulting Carnot-Carathéodory metric is left-invariant.

Convention. Whenever we speak of the metric space properties of a Carnot group, this refers to the properties of the Carnot group equipped with a Carnot-Carathéodory metric.

The topology induced by a Carnot-Carathéodory metric coincides with the topology of the underlying manifold. However, there are aspects which are not captured by Riemannian geometric considerations. The right tool to study these is the Hausdorff measure. For example, it can be shown that the Hausdorff dimension is given by $p = \sum_{i=1}^{n} i \dim V^i$, which is usually different than the dimension of the manifold [Mit85, Theorem 2]. This is related to the self-similarity properties of these metric spaces, which is visible in the fact that every ball can be covered by smaller copies of itself. Otherwise said, it is true that there exist positive constants $\sigma \leq \tau$, that may depend on x, such that for the ball $B(x, r)$ centred at x with radius r we have

$$\sigma r^p \leq \mathscr{H}^p(B(x, r)) \leq \tau r^p, \tag{2.2}$$

where p is the Hausdorff dimension of the Carnot group [Gro96, Section 0.6]

In any metric space, we can consider transformations that change the sizes of objects, but leave their relative properties invariant. These transformations are called homotheties. In a Carnot group, we would like to additionally impose that homotheties are group homomorphisms. We will see that for a Carnot group with any Carnot-Carathéodory distance these transformations exist, and they will be important for introducing differentiability on Carnot groups.

Definition 2.8. Let N be a metric space with distance function d. A *homothety of ratio a* is an automorphism $\delta_a \colon N \to N$ that scales distances by a factor a, that is, for all $p, q \in N$, we have

$$d(\delta_a p, \delta_a q) = a d(p, q).$$

Lemma 2.9. *For a Carnot group with derivation α the automorphism $e^{t\alpha}$ is a homothety of ratio e^t for any Carnot-Carathéodory metric d.*

Proof. This is essentially a consequence of the horizontality of distance-measuring curves and the fact that $\alpha|_{V^1} = I_{\dim V^1}$. Recall that γ is a horizontal curve if $\dot\gamma(s) \in d_e L_{\gamma(s)} V^1$, hence $\alpha\dot\gamma(s) = \dot\gamma(s)$, so that $\frac{d}{ds}\left(e^{t\alpha}\gamma(s)\right) = e^t\dot\gamma(s)$. Let g_e be the inner product on V^1 with respect to which the Carnot-Carathéodory metric is defined. The simple calculation

$$g_{e^{t\alpha}\gamma(s)}\left(\frac{d}{ds}\Big(e^{t\alpha}\gamma(s)\Big), \frac{d}{ds}\Big(e^{t\alpha}\gamma(s)\Big)\right) = e^t g_{\gamma(s)}\left(\frac{d}{ds}\gamma(s), \frac{d}{ds}\gamma(s)\right),$$

together with the observation that if γ is a horizontal curve from p to q, then $e^{t\alpha}\gamma$ is a horizontal curve from $e^{t\alpha}p$ to $e^{t\alpha}q$, shows that

$$\begin{aligned} d(e^{t\alpha}p, e^{t\alpha}q) &= \inf\Big\{\text{length}(e^{t\alpha}\gamma)\colon e^{t\alpha}\gamma \text{ horizontal from } e^{t\alpha}p \text{ to } e^{t\alpha}q\Big\} \\ &= \inf\left\{\int \left(g_{e^{t\alpha}\gamma(s)}\left(\frac{d}{ds}\left(e^{t\alpha}\gamma(s)\right), \frac{d}{ds}\left(e^{t\alpha}\gamma(s)\right)\right)\right)^{1/2} ds\colon e^{t\alpha}\gamma \text{ horizontal from } e^{t\alpha}p \text{ to } e^{t\alpha}q\right\} \\ &= \inf\left\{e^t \int \left(g_{\gamma(s)}\left(\frac{d}{ds}\gamma(s), \frac{d}{ds}\gamma(s)\right)\right)^{1/2} ds\colon \gamma \text{ horizontal from } p \text{ to } q\right\} \\ &= e^t d(p,q), \end{aligned}$$

as desired. □

2.3 Differentiability on Carnot groups

To motivate Pansu's definition of differentiability on Carnot groups, let us first recall differentiability of functions between real vector spaces.

Let $U \subseteq \mathbb{R}^n$ be open. A function $f\colon U \to \mathbb{R}^m$ is differentiable in $x \in U$ if for all $v \in \mathbb{R}^n$ the limit

$$d_v f(x) := \lim_{h\to 0} \frac{f(x+hv) - f(x)}{h}$$

exists and the map $df(x)$ defined by $v \mapsto d_v f(x)$ is linear. We call $df(x)$ the differential of f at x.

For functions between Carnot groups, we can now generalise this concept by replacing addition with group multiplication, and the stretching by h with applying a homothety $e^{-t\alpha}$. The limit $h \to 0$ is replaced by $t \to \infty$, so that the ratio e^{-t} of the homothety tends to 0. This leads to the following definition.

Definition 2.10. Let N, N' be Carnot groups and let $U \subseteq N$ be open. For a continuous map $f\colon U \to N'$ we say that f is *differentiable* at $x \in U$ if

$$Df(x) = \lim_{t\to\infty} e^{t\alpha'} \circ L^{-1}_{f(x)} \circ f \circ L_x \circ e^{-t\alpha}$$

converges uniformly on compact sets. This limit is a group homomorphism and we call it the *differential of f at x*. In analogy to the ordinary derivative, for $\mu \in N$ we define the *differential in the direction μ* as $D_\mu f(x) = \lim_{t\to\infty} e^{t\alpha'} \circ L_{f(x)}^{-1} \circ f \circ L_x \circ e^{-t\alpha}(\mu)$.

From now on, whenever we speak of differentiability, we will refer to this definition, and the few times that we invoke the classical notion of differentiability, we will emphasise this by using the attribute *ordinary*. For additional clarity, we denote the ordinary differential of f at x by $df(x)$ whereas the differential in the sense introduced above is written with a capital letter D.

Note that if we take N, N' to be the additive groups $\mathbb{R}^n, \mathbb{R}^m$ respectively with $\alpha = id_{\mathbb{R}^n}$, $\alpha' = id_{\mathbb{R}^m}$, then for $v \in \mathbb{R}^n$ we have

$$D_v f(0) = \lim_{t\to\infty} e^{t\alpha'} \circ f \circ e^{-t\alpha}(v) = \lim_{t\to\infty} \frac{f(0 + e^{-t}v) - f(0)}{e^{-t}},$$

which coincides with the ordinary differential $d_v f(0)$.

3 Differentiability of quasiconformal homeomorphisms

3.1 Outline of the proof

The goal of this section is to prove that quasiconformal homeomorphisms between open subsets of Carnot groups are differentiable. Our proof will also imply that Lipschitz functions between Carnot groups are differentiable. The main result of this section is the following proposition.

Proposition 3.1. *Let N, N' be Carnot groups, and let $U \subseteq N$ and $U' \subseteq N'$ be open. Any Lipschitz map and any quasiconformal homeomorphism from U to U' is almost everywhere differentiable, and the differential is a homomorphism of Carnot groups. If $N = N'$, then the differential commutes with the homotheties $e^{t\alpha}$.*

In Section 3.2, we prove that if the differential exists almost everywhere in the direction μ, then it also exists almost everywhere in the direction $e^{a\alpha}\mu$ for $a \in \mathbb{R}$, and if it exists in two directions, say μ and ν, then it also exists in the product of the directions $\mu\nu$. Moreover, these operations are compatible with the homotheties and the group multiplication. This preliminary result will allow us to deduce two things. First, it allows to reduce the differentiability of maps to the differentiability of curves, and second, it implies that the differential, if it exists, is a group homomorphism.

To see why this allows a reduction of our considerations to the case of curves, we recall from Section 2.1 that a Carnot group can globally be parameterised with exponential coordinates, that the Baker-Campbell-Hausdorff series is finite and polynomial, and that the subspace V^1 of a gradation of its Lie algebra generates the Lie algebra. Given a basis $X_1, \cdots, X_r$ of V^1, the curves

$$\{s \mapsto \exp(sX_i) \colon i = 1, \ldots, r\}$$

generate the Carnot group. These curves are clearly horizontal. If restricted to a bounded domain, they have finite length and hence are rectifiable.

In Section 3.3 we prove that any rectifiable curve in a Carnot group is differentiable. The proof of the main result of this section then follows by showing that quasiconformal homeomorphisms map rectifiable curves to rectifiable curves. This is done in Section 3.4. In particular, if f is a quasiconformal homeomorphism, then the curves

$$s \mapsto f(\exp(sX_i)), \qquad i = 1, \ldots, r,$$

are rectifiable and thus differentiable. We can then repeatedly apply the preliminary result to deduce differentiability of f. We will largely follow Pansu's original proof, with the exception that Pansu's analysis of rectifiable curves is replaced by the shorter proof of [LD23] and [Mon01].

3.2 Reduction to the case of curves

The first step in proving Proposition 3.1 is a reduction to the case of curves. We will show that if the differential exists almost everywhere in two directions μ and ν, then it also exists in the direction $e^{a\alpha}\mu e^{b\alpha}\nu$ for $a, b \in \mathbb{R}$. The proof of this statement is divided into two parts. In Lemma 3.5 we show that the existence of $D_\mu f(x)$ almost everywhere implies the existence of $D_{e^{a\alpha}\mu} f(x)$ almost everywhere, and in Lemma 3.7 we treat the product of two directions. Before we prove Proposition 3.1, we recall a few general results that we will use to prove the main result of this section.

Lemma 3.2. [Rud87, p.55 and p.73] *Let (X, μ) be a measure space with $\mu(X) < \infty$ and let Y be a separable metric space. Let $(f_t)_{t>0}$ be a family of measurable functions from X to Y depending on a real parameter $t \in (0, \infty)$. Suppose that $(f_t)_t$ converges almost everywhere pointwise to f as $t \to 0$. Then for every $\tau > 0$, there exists a measurable subset $K \subseteq X$ such that $\mu(X \setminus K) < \tau$ and $(f_t)_t$ converges to f uniformly on K.*

We point out that open subsets U of Carnot groups equipped with a Carnot-Carathéodory metric and Hausdorff measure $\mathscr{H}^p$, where $\mathscr{H}^p(U) < \infty$, satisfy all requirements of this proposition.

Convention. Whenever we consider the measure $\mathscr{H}^p$, we may take p to be the Hausdorff dimension of the Carnot group unless otherwise specified.

Definition 3.3. Let X be a space equipped with a measure μ, and $A \subseteq X$ a measurable subset. A *μ-density point of A* is a point $x \in A$ satisfying

$$\lim_{r\to 0}\frac{\mu(A\cap B(x,r))}{\mu(B(x,r))} = 1.$$

Equivalently, as measures are countably additive, one can require $\lim_{r\to 0}\frac{\mu(A^c\cap B(x,r))}{\mu(B(x,r))} = 0$.

Measurable sets in Carnot groups equipped with the Hausdorff measure also have the following useful property.

Lemma 3.4. [Mon01, Theorem 1.6.5]*. If N is a Carnot group equipped with a Carnot-Carathéodory metric and Hausdorff measure $\mathscr{H}^p$, and K is a measurable set in N, then $\mathscr{H}^p$-almost every point of K has density 1.*

We first observe that the differential is compatible with homotheties.

Lemma 3.5. *Let N, N' be Carnot groups and let $U \subseteq N$ be open. Consider $f\colon U \to N'$, and let $\mu \in N$ and $x \in U$. If the differential $D_\mu f(x)$ exists, then $D_{e^{a\alpha}\mu} f(x)$ exists for all $a \in \mathbb{R}$ and $D_{e^{a\alpha}\mu} f(x) = e^{a\alpha'} D_\mu f(x)$.*

Proof. We assume that for $\mu \in N$ and $x \in U$ the limit

$$D_\mu f(x) = \lim_{t\to\infty} e^{t\alpha'} \circ L_{f(x)}^{-1} \circ f \circ L_x \circ e^{-t\alpha}(\mu)$$

exists. Then

$$\begin{aligned}
D_{e^{a\alpha}\mu} f(x) &= \lim_{t\to\infty} e^{t\alpha'} \circ L_{f(x)}^{-1} \circ f \circ L_x \circ e^{-t\alpha}(e^{a\alpha}\mu)\\
&= \lim_{t\to\infty} e^{a\alpha'} \circ e^{(t-a)\alpha'} \circ L_{f(x)}^{-1} \circ f \circ L_x \circ e^{-(t-a)\alpha}(\mu)\\
&= \lim_{s:=(t-a)\to\infty} e^{a\alpha'} \circ e^{s\alpha'} \circ L_{f(x)}^{-1} \circ f \circ L_x \circ e^{-s\alpha}(\mu)\\
&= e^{a\alpha'}\left(\lim_{s\to\infty} e^{s\alpha'} \circ L_{f(x)}^{-1} \circ f \circ L_x \circ e^{-s\alpha}(\mu)\right)\\
&= e^{a\alpha'} D_\mu f(x),
\end{aligned}$$

where for obtaining the last equality we exchanged the continuous map $e^{a\alpha'}$ and the limit. □

In particular, this implies that the differential of a map from a Carnot group to itself commutes with the homotheties $e^{a\alpha}$. We now want to prove that if the differential in two directions exists almost everywhere, then it exists also in the product of the two directions, and the latter is given by the product of the differentials in the two directions. As in the classical setting, our results on differentiability will only be applicable to functions that locally do not vary too much. To make this precise, we introduce the following.

Definition 3.6. Let X, X' be metric spaces, let f be a map from X to X'. The *local dilation* of f, denoted Lip_f, at the point $x \in X$ is defined as

$$\mathrm{Lip}_f(x) = \limsup_{y \to x} \frac{d(f(x), f(y))}{d(x, y)}.$$

We can now formulate and prove the second part of our statement above.

Lemma 3.7. *Let N, N' be Carnot groups, and let $U \subseteq N$ be open. Let $f\colon U \to N'$ be a map whose local dilation Lip_f is finite almost everywhere. Let $\mu, \nu \in N$. Assume that for almost every $x \in N$, the limits*

$$D_\mu f(x) = \lim_{t\to\infty} e^{t\alpha'}(f(x)^{-1} f(xe^{-t\alpha}\mu)),$$
$$D_\nu f(x) = \lim_{t\to\infty} e^{t\alpha'}(f(x)^{-1} f(xe^{-t\alpha}\nu))$$

exist. Then for every ω of the form $\omega = e^{a\alpha}\mu e^{b\alpha}\nu$, where $a, b \in \mathbb{R}$, and for almost every $x \in N$, the limit $D_\omega f(x) = \lim_{t\to\infty} e^{t\alpha'}(f(x)^{-1} f(xe^{-t\alpha}\omega))$ exists and is equal to $e^{a\alpha'} D_\mu f(x) e^{b\alpha'} D_\nu f(x)$.

Proof. First we note that with Lemma 3.5 it is enough to consider the case $\omega = \mu\nu$, where we are renaming $\nu \to e^{-b\alpha}\nu$. Without loss of generality, we may assume $\mathscr{H}^p(U) < \infty$. Setting

$$f_t(x, \nu) := e^{t\alpha'}(f(x)^{-1} f(xe^{-t\alpha}\nu)), \tag{3.1}$$

we note that for all $t \in \mathbb{R}$ and $x \in U$ the function $f_t(\cdot, \nu)$ is measurable because we assumed that f is locally Lipschitz. Per assumption we have that $f_t(x, \nu) \to D_\nu(x)$ as $t \to \infty$ for almost every $x \in U$ and fixed $\nu \in N$. Lemma 3.2 implies that we can find a closed subset $F \subseteq U$ whose complement has measure $\mathscr{H}^p(U \setminus K) < \tau$ for arbitrary $\tau > 0$, and

(a) $D_\mu f(x)$ and $D_\nu f(x)$ exist for all $x \in F$ and are the respective limits of $f_t(x, \mu)$ and $f_t(x, \nu)$,

(b) $x \mapsto D_\nu f(x)$ is continuous on F,

(c) $e^{t\alpha'}(f(x)^{-1} f(xe^{-t\alpha}\nu)) \to D_\nu f(x)$ uniformly in x on F.

The statement (a) holds after removing from F a suitable zero-measure subset, (b) is true because $x \to f_t(x,\nu)$ continuous for all t and the uniform limit of a family of continuous functions on the compact set F is continuous on F, and (c) is a consequence of the fact that $f_t(\cdot,\nu)$ is continuous for all $t \in \mathbb{R}$, so that the uniform limit $D_\nu f(\cdot)$ is continuous.

We will show that this proves the proposition for those $x \in F$ such that $xe^{t\alpha}\mu \in F$ for all $t \in \mathbb{R}$. Assuming $xe^{t\alpha}\mu \in F$, we decompose $e^{t\alpha'}(f(x)^{-1}f(xe^{-t\alpha}\omega))$ into a product of three terms and investigate the convergence of each of the factors. Note that $e^{-t\alpha}(\mu\nu) = e^{-t\alpha}\mu e^{-t\alpha}\nu$ and write

$$\begin{aligned} e^{t\alpha'}(f(x)^{-1}f(xe^{-t\alpha}\omega)) &= e^{t\alpha'}(f(x)^{-1}f(xe^{-t\alpha}(\mu\nu))) \\ &= \underbrace{e^{t\alpha'}\left(f(x)^{-1}f(xe^{-t\alpha}\mu)\right)}_{=:(1)}\underbrace{e^{t\alpha'}\left(f(xe^{-t\alpha}\mu)^{-1}f(xe^{-t\alpha}\mu e^{-t\alpha}\nu)\right)(D_\nu f(xe^{-t\alpha}\mu))^{-1}}_{=:(2)}\underbrace{D_\nu f(xe^{-t\alpha}\mu)}_{=:(3)}, \end{aligned}$$

where (1) converges to $D_\mu f(x)$ by (a), (2) converges to 1 by (c), and (3) converges to $D_\nu f(x)$ by (b), considering that $\lim_{t\to\infty} e^{-t\alpha}\mu = 1$ as $t \to \infty$. The union of such sets K with $\tau^{-1} \in \mathbb{N}$ is a full measure set. In the case that $xe^{t\alpha}\mu \in F$ for all t, this concludes the proof.

The requirement that $xe^{t\alpha}\mu \in F$ for all t is not necessarily satisfied, but it can be shown that, if x is a $\mathscr{H}^p$-density point of F, then there is a point of F that is close to $xe^{t\alpha}\mu$. Let λ_t be the distance from $xe^{t\alpha}\mu$ to F, and let μ'_t be such that $xe^{t\alpha}\mu'_t$ realises that distance. As $B(xe^{t\alpha}\mu,\lambda_t) \cap F$ is an $\mathscr{H}^p$-zero set,

$$\frac{\mathscr{H}^p(B(x,e^{-t}+\lambda_t)\setminus F)}{\mathscr{H}^p(B(x,e^{-t}+\lambda_t))} \geq \frac{\mathscr{H}^p(B(xe^{-t\alpha}\mu,\lambda_t))}{\mathscr{H}^p(B(x,e^{-t}+\lambda_t))} = \left(\frac{\lambda_t}{e^{-t}+\lambda_t}\right)^p.$$

The left side tends to 0 as t tends to ∞ because x is a $\mathscr{H}^p$-density point of F. Thus $e^t\lambda_t$ tends to 0 when t tends to ∞, and μ'_t tends to μ when t tends to ∞. We write $e^{t\alpha'}f(x)^{-1}f(xe^{t\alpha}\omega)$ as

$$e^{t\alpha'}f(x)^{-1}f(xe^{-t\alpha}\omega) = (1)(2)(3)(4)(5),$$

where we set

$$\begin{aligned} (1) &= e^{t\alpha'}\left(f(x)^{-1}f(xe^{-t\alpha}\mu)\right) \overset{\text{by (a)}}{\longrightarrow} D_\mu f(x), \\ (2) &= e^{t\alpha'}\left(f(xe^{-t\alpha}\mu)^{-1}f(xe^{-t\alpha}\mu'_t)\right) \overset{\text{by (*) below}}{\longrightarrow} 1, \\ (3) &= e^{t\alpha'}\left(f(xe^{-t\alpha}\mu'_t)^{-1}f(xe^{t\alpha}\mu'_t e^{-t\alpha}\nu)\right)\left(D_\nu f(xe^{t\alpha}\mu'_t)\right)^{-1} \overset{\text{by (c)}}{\longrightarrow} 1, \\ (4) &= D_\nu f(xe^{t\alpha}\mu'_t), \\ (5) &= e^{t\alpha'}\left(f(xe^{-t\alpha}\mu'_t e^{-t\alpha}\nu)^{-1}f(xe^{-t\alpha}\mu e^{-t\alpha}\nu)\right) \overset{\text{by (**) below}}{\longrightarrow} 1. \end{aligned}$$

It remains to justify the convergence and limits of (2) and (5). Let d, d' denote the left-invariant Carnot-Carathéodory distances in N, N' respectively. Then we have

$$\begin{aligned}
d'((5),1) &= d'\left(e^{t\alpha'}\left(f(xe^{-t\alpha}\mu_t' e^{-t\alpha}\nu)^{-1} f(xe^{-t\alpha}\mu e^{-t\alpha}\nu)\right),1\right) \qquad (**)\\
&= d'\left(e^{t\alpha'}\left(f(xe^{-t\alpha}\mu_t' e^{-t\alpha}\nu)^{-1}\right) e^{t\alpha'}\left(f(xe^{-t\alpha}\mu e^{-t\alpha}\nu)\right),1\right)\\
&= d'\left(e^{t\alpha'}\left(f(xe^{-t\alpha}\mu e^{-t\alpha}\nu)\right), e^{t\alpha'}\left(f(xe^{-t\alpha}\mu_t' e^{-t\alpha}\nu)\right)\right)\\
&= d'\left(f(xe^{-t\alpha}\mu e^{-t\alpha}\nu), f(xe^{-t\alpha}\mu_t' e^{-t\alpha}\nu)\right)\\
&\leq Me^t d\left(e^{-t\alpha}(\mu\nu), e^{-t\alpha}(\mu_t'\nu)\right)\\
&= Md\left(\mu\nu, \mu_t'\nu\right) \to 0,
\end{aligned}$$

where the arguments from line to line are that $e^{t\alpha'}$ is a group homomorphism, then we use the left-invariance of the distance function, then the fact that $e^{t\alpha'}$ is a homothety of ratio e^t, then that we assume that Lip_f is globally bounded by M, an assumption that we justify below, then again that $e^{-t\alpha}$ is a homothety of ratio e^{-t}, and finally the convergence to 0 as $t \to \infty$ follows from $\mu_t' \to \mu$ as established earlier, together with the fact that right-multiplication is continuous.

Similarly, one shows that

$$d'((2),1) = d'\left(e^{t\alpha'}\left(f(xe^{-t\alpha}\mu)^{-1} f(xe^{-t\alpha}\mu_t')\right),1\right) \leq Md(\mu,\mu_t') \to 0. \qquad (*)$$

If Lip_f were bounded everywhere on F as we assumed in $(*)$ and $(**)$, then we would have proven the existence of $D_{\mu\nu}f(x)$ at almost every $\mathscr{H}^p$-density point of F. By Lemma 3.4, this is almost everywhere of F. However, we have boundedness of Lip_f only almost everywhere on U. Therefore, to conclude the general statement we need to take a subset of F on which $\mathrm{Lip}_f \leq M$. According to our assumption, almost every point x lies in one of the sets

$$A_k = \left\{ x \in N\colon \text{ for all } y \in B(x,k^{-1}) \text{ we have } \frac{d'(f(x),f(y))}{d(x,y)} \leq k \right\}.$$

Taking $\bigcup_{k\leq K_0} A_k \cap F$ for some $K_0 \in \mathbb{N}$ provides us with a subset of F on which f is Lipschitz again. This choice can be made such that its complement has arbitrarily small measure. Taking the union over such sets with increasingly small measure finishes the proof. □

As outlined earlier, the two previous lemmas can be used to reduce the proof of Proposition 3.1 to the differentiability of curves. This is because, due to the bijectivity of the exponential map, every $\mu \in N$ can be written as

$$\mu = \exp\left(\sum_{i=1}^{n} x_i X_i\right), \tag{3.2}$$

where $\{X_1, \cdots, X_n\}$ is a basis of the Lie algebra $\mathfrak{n}$ of N, and $x_i \in \mathbb{R}$. Moreover, $\mathfrak{n}$ is generated by iterated brackets of elements of V^1, and the Baker-Campbell-Hausdorff series is polynomial and finite. We can rewrite (3.2) as

$$\mu = \prod_{i=1}^{m} \exp(s_i(x_1, \cdots, x_n)X_i),$$

for some $m \in \mathbb{N}$, where now $X_i \in \{X_1, \cdots, X_r\}$, which we take to be a basis of V^1.

By Lemma 3.7, if $D_{\exp(s_1X_1)}f(x)$ and $D_{\exp(s_2X_2)}f(x)$ exist almost everywhere, then the differential on the product of the directions $D_{\exp(s_1X_1)\exp(s_2X_2)}f(x)$ exists almost everywhere. It is easy to see that this can be iterated to conclude the existence of $D_\mu f(x)$.

Corollary 3.8. *Let N, N' be Carnot groups, and let $U \subseteq N$ be an open subset. Let f be an application from U into N' such that $\mathrm{Lip}_f < \infty$ almost everywhere. Let $\mathfrak{n}$ be the Lie algebra of N and let $\{X_1, \cdots, X_r\}$ generate $\mathfrak{n}$. If for all $i \in \{1, \cdots, r\}$ and for almost every $x \in N$ the curve*

$$s \mapsto f(x \exp(sX_i))$$

is almost everywhere differentiable, then f is differentiable almost everywhere, and the differential $\mu \mapsto D_\mu f(x)$ is a group homomorphism.

Proof. In Lemma 3.7 it was established that $\mu \mapsto D_\mu f(x)$ is a group homomorphism. We further know that for Carnot groups the exponential map is a diffeomorphism, this has been shown in Lemma 2.3. As V^1 generates the Lie algebra $\mathfrak{n}$, the set of curves $\{\exp(sX_i) \colon s \in \mathbb{R}, i = 1, \cdots, r\}$ generates N. Applying Lemma 3.7 yields the existence of the differential $D_\mu f(x)$ for all $\mu \in N$.

It remains to confirm that the convergence in μ of $\lim_{t\to\infty} e^{t\alpha'}(f(x)^{-1}f(xe^{-t\alpha}\mu))$ to $D_\mu(x)$ for fixed x is uniform. Note that in the proof of Lemma 3.7 we had fixed μ and investigated the convergence of the expression.

We define f_t as in (3.1), and the proper submersion

$$\tilde{\mu} \colon \mathbb{R}^m \to N,\ (a_1, \cdots, a_m) \mapsto \prod_{i=1}^{m} e^{a_i\alpha} \exp(X_i),$$

where $X_i \in \{X_1, \cdots, X_r\}$. This will be used to parameterise the curves, one by one in each argument. Inserting $\tilde{\mu}$ into $f_t(x, \cdot)$, we get

$$f_t(x, \tilde{\mu}(a_1, \cdots, a_m)) = f_t(x, \prod_{i=1}^{m} e^{a_i \alpha} \exp(X_i)).$$

This can inductively be decomposed into a product of terms $f_t(x, e^{a_j \alpha} X_j)$, as in the proof of Lemma 3.7, that converge to $e^{a_j \alpha'} D_{\exp(X_j)}$ by assumption. In general, additional terms appear in this decomposition, but it is easy to see that these converge to 1. From Lemma 3.7, we know that $f_t(x, \tilde{\mu}(a_1, \cdots, a_m))$ has a limit which is the product of the limits $e^{a_j \alpha'} D_{\exp X_j}$. The fact that this convergence is uniform in t follows from the fact that for each of the terms, we see from the proof of Lemma 3.7 that the convergence only depends on $|a_j|$ and on the convergence of $f(\exp(X_j))$ to $D_{\exp(X_j)}(x)$. This observation completes the proof. □

3.3 Differentiability of rectifiable curves

Corollary 3.8 shows that we only need differentiability of the curves $s \mapsto f(x \exp(sX_i))$ almost everywhere, where $X_i \in \{X_1, \cdots, X_r\}$ generate the Lie algebra, to conclude differentiability of f almost everywhere. For using this to prove that quasiconformal homeomorphisms are differentiable, we introduce rectifiable curves.

Definition 3.9. Given a path p in a metric space X, we define its length as follows. A partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

of the interval $[a, b]$ defines a finite collection of points $p(t_0), p(t_1), \cdots, p(t_n)$ in the space X. The length of p is then defined to be

$$\text{length}(p) = \sup_{a=t_0<t_1<\cdots<t_n=b} \sum_{i=0}^{n-1} \text{dist}(p(t_i), p(t_{i+1})),$$

where the supremum is taken over all possible partitions of $[a, b]$ and all integers n.

If the length of p is finite, then p is called *rectifiable*.

We remark that in a Carnot group with Carnot-Carathéodory spaces, this coincides with the notion of lengths of curves that we introduced earlier. Clearly the curves

$$\gamma \colon [a, b] \to G, \quad s \mapsto x \exp(sX_i) \qquad \text{for } a, b \in \mathbb{R} \text{ and } X_i \in V^1$$

are rectifiable, with length $(b - a) \|X_i\|$. We will later show that quasiconformal maps map rectifiable curves to rectifiable curves, in particular, the curves $s \mapsto f(\exp(sX_i))$ for $i = 1, \cdots, r$, are rectifiable. In this section, we prove that rectifiable curves are differentiable, so that the curves

$s \mapsto f(\exp(sX_i))$ are differentiable. With Corollary 3.8 the differentiability of quasiconformal homeomorphism follows.

To show the differentiability of rectifiable curves, we will make use of the Lebesgue differentiation theorem in the form stated below.

Lemma 3.10. [Rud87, Theorem 7.7] *For $f \in L^1(\mathbb{R}^k)$, a point $x \in \mathbb{R}^k$ for which*

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)-f(x)|\,dy = 0$$

is called a Lebesgue point *of f. Almost every point $x \in \mathbb{R}^k$ is a Lebesgue point of f.*

We observe that rectifiable curves are horizontal almost everywhere.

Lemma 3.11. *Let $\gamma\colon I \to N$ be a rectifiable curve in a Carnot group N, where I is some interval in $\mathbb{R}$. If $\dot\gamma$ denotes the ordinary differential of γ, then we have $\dot\gamma \in V^1$ almost everywhere.*

Proof. Without loss of generality we may assume $I = [0,1]$. By bijectivity of the exponential map, there exists a curve $\sigma\colon I \to \mathfrak{n}$ such that $\gamma(s) = \exp(\sigma(s))$, and we can assume $\sigma(0) = 0$. Choose a subdivision Σ of $[0,s]$. We approximate σ up to a value s of the curve parameter by a piecewise linear curve which meets σ at the points $\{\sigma(t_i)\colon t_i \in \Sigma\}$. The corresponding approximation of γ is given by $c_\Sigma(s) = \prod_{k=0}^{N-1} \exp(\sigma(t_{k+1}) - \sigma(t_k))$. Choose a subdivision Σ of $[0,s]$. We consider the curve $c(s) = \lim_{|\Sigma|\to 0} c_\Sigma(s)$.

If σ is parameterised by arc length, then by [Pan83, Lemma 42], it holds that

$$\|\sigma(s) - \log c(s)\| \leq \text{const.}s^2. \tag{3.3}$$

Any $v \in \mathfrak{n}$ can be written as $v = \sum_{i=1}^{n} v_i$, where $v_i \in V^i$. Using the dilations of the Carnot group, we see that

$$\left\|v^2 + \cdots + v^n\right\| \leq \text{const.}d(1,\exp(v))^2. \tag{3.4}$$

We assume that c is parameterised by arc length and set

$$c(s+t) = c(s)\exp(\sigma^1(t) + \cdots + \sigma^n(t)).$$

Combining (3.3) and (3.4), we conclude

$$\sigma(s+t) - \sigma = \sigma^1(s) + \mathcal{O}(s^2).$$

This shows that γ is almost everywhere tangent to V^1 and that $\dot\gamma = \sigma^1$. □

Our result about the differentiability of rectifiable curves is the following proposition. We remark that our proof of this partial result does not follow the original proof given by Pansu in [Pan89b], but instead is close to [LD23, Proposition 8.6.6] and [Mon01, Lemma 2.1.4]. Both rely on the same ideas as Pansu's proof, but are considerably shorter.

Proposition 3.12. *Let I be an interval in $\mathbb{R}$. A locally rectifiable curve $\gamma\colon I \to N$ is almost everywhere differentiable. If $\dot\gamma(t) \in T_{\gamma(t)}N$ is the ordinary derivative, then $\dot\gamma(t) \in d_e L_{\gamma(t)}(V_1)$ and the derivative is expressed as $D_\mu\gamma(t) = exp(\mu\dot\gamma(t))$, where $\mu \in N$.*

Proof. We may take $I = [0,1]$. Further, we assume that $\mu = 1$, because the other cases can be derived from this using left-multiplication. Let $X_1, \cdots, X_r$ be a basis of V^1. By Lemma 3.11 there exist functions $h_1, \cdots, h_r \in L^\infty([0,1];\mathbb{R})$ such that for almost all $t \in [0,1]$ we can write

$$\dot\gamma(t) = \sum_{j=1}^{r} h_j(t) X_j(\gamma(t)). \tag{3.5}$$

As γ is locally rectifiable, it is L-Lipschitz, so that we may take $|h_j(t)| \leq L$ for all t. By Lemma 3.10, almost every point in I is a Lebesgue point for each of the functions h_j, and we choose $x \in [0,1]$ both a point at which the ordinary differential of γ exists and a Lebesgue point for all h_j, that is,

$$\frac{1}{|t-x|}\int_x^t |h_j(s) - h_j(t)|\, ds \to 0 \quad \text{ as } t \to x.$$

Up to replacing γ with the curve $t \mapsto \gamma(x)^{-1}\gamma(t+x)$ we may also assume that $x = 0$ and $\gamma(x) = 1$. Our aim is to show that

$$\lim_{t\to 0} \delta_{1/t}\gamma(t) = \exp(\dot\gamma(0)).$$

Note that we have $\dot\gamma(0) = \sum_{j=1}^{r} h_j(0)X_j(0)$, since 0 is a Lebesgue point for all h_j.

We set $\eta_t(s) := \delta_{1/t}\gamma(ts)$. Then each $\eta_t\colon [0,1] \to N$ is a curve starting at 0 that is L-Lipschitz,

$$d(\eta_t(s), \eta_t(s')) = d(\delta_{1/t}\gamma(ts), \delta_{1/t}\gamma(ts')) \leq \frac{L}{t}\,|ts - ts'| = L\,|s - s'|\,.$$

We may thus apply the Arzelà-Ascoli theorem to conclude that every sequence $(\eta_{t_k})_{k\in\mathbb{N}}$ has a uniformly converging subsequence. Using (3.5), we calculate

$$\dot\eta_t(s) = \log\left(\frac{d}{ds}\delta_{1/t}\gamma(ts)\right) = \log\Big((\delta_{1/t})_*(t\dot\gamma(ts))\Big) = \dot\gamma(ts) = \sum_{j=1}^{r} h_j(ts)X_j(\eta_t(s)). \tag{3.6}$$

Our next goal is to prove that η_t uniformly converges to η_0 as $t \to 0$. This will complete the proof since in particular, $\eta_t(1) = \delta_{1/t}\gamma(t) \to \exp(\dot\gamma(0))$. To prove the claim, we invoke the Arzelà-Ascoli theorem to obtain for every sequence $(t_k)_{k\in\mathbb{N}}$ converging to 0 a further subsequence t_{k_i} and a curve $\xi\colon [0,1] \to N$ to which the sequence $(\eta_{t_{k_i}})_{i\in\mathbb{N}}$ converges uniformly. We want to show that this limit ξ satisfies

$$\dot\xi(s) = \sum_{j=1}^{r} h_j(0) X_j(\xi(s))$$

for almost every $s \in [0,1]$, because this will imply that $\eta_{t_{k_i}} \to \exp(\dot\gamma(0))$.

Let us introduce an auxiliary curve σ as the curve in N with $\sigma(0) = 1$ and $\dot\sigma(s) = \sum_{j=1}^{r} h_j(0)X_j(\xi(s))$. If we can prove that $\eta_{t_{k_i}}$ converges pointwise to σ, we are done. To obtain $\log(\sigma(v)\eta_{t_{k_i}}(v)^{-1})$, we integrate $\dot\sigma(s) - \dot\eta_{t_{k_i}}(s)$ from 0 to an arbitrary $v \in (0,1)$, inserting $\dot\eta_{t_{k_i}}(s)$ from (3.6). This yields

$$\begin{aligned}\log(\sigma(v)\eta_{t_{k_i}}(v)^{-1}) &= \sum_{j=1}^{r} \int_0^v \Big(h_j(0)X_j(\xi(s)) - h_j(t_{k_i}s)X_j(\eta_{t_{k_i}}(s))\Big)\,ds \\ &\le \sum_{j=1}^{r} \int_0^v \Big(|h_j(0) - h_j(t_{k_i})|\, X_j(\xi(s)) + |h_j(t_{k_i}s)|\, \Big|X_j(\xi(s)) - X_j(\eta_{t_{k_i}}(s))\Big|\Big)\,ds.\end{aligned}$$

By continuity of X_j, the last summand tends to 0 as $i \to \infty$, and for the first summand we have

$$\sum_{j=1}^{r} \int_0^v |h_j(0) - h_j(t_{k_i}s)|\,ds \le \sum_{j=1}^{r} \int_0^1 |h_j(0) - h_j(t_{k_i}s)|\,ds = \sum_{j=1}^{r} \frac{1}{t_{k_i}} \int_0^{t_{k_i}} |h_j(0) - h_j(u)|\,du,$$

which tends to 0 as $i \to \infty$, or equivalently $t_{k_i} \to 0$, since 0 is a Lebesgue point for every h_j. It follows that η_t converges uniformly to η_0, where $\eta_0(s) = \delta_{1/t}\gamma(ts)|_{t=0}$, and we conclude that $\delta_{1/t}\gamma(t) = \eta_t(1) \to \exp(\dot\gamma(0))$ as $t \to 0$, as desired. □

3.4 Differentiability of quasiconformal homeomorphisms

In this section, we prove that quasiconformal homeomorphisms map rectifiable curves to rectifiable curves. Using Corollary 3.8 and Proposition 3.12, this will conclude the proof that quasiconformal homeomorphisms are differentiable almost everywhere. We start with some definitions.

Notation. When $B = B(x,r)$ is a ball in a metric space, we write kB for the ball $B(x,kr)$.

Definition 3.13. Let X be a metric space. An *annulus* in X is a tuple of subsets $(a,\tilde a)$, where $a, \tilde a \subseteq X$ and $a \subseteq \tilde a$. It is a k-*annulus* if there exists B such that

$$B \subseteq a \subseteq \tilde a \subseteq kB.$$

Definition 3.14. Let X and X' be metric spaces and let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism. A map $f\colon X \to X$ is called η-*quasisymmetric* if it sends any k-annulus that is small enough in X into an $\eta(k)$-annulus of X. A homeomorphism f between X and X' is called *quasiconformal* if f and f^{-1} are η-quasisymmetric.

In the course of proving that quasiconformal homeomorphisms map rectifiable curves to rectifiable curves, we apply the following lemma several times.

Lemma 3.15. [Fed69, Theorem 2.8.4] *Let X be a metric space. Let $\{B_i\colon i \in I\}$ be a cover of X with balls. Then there exists a subfamily $\{B_j\colon j \in J\}$ of disjoint balls such that the balls with the same centre $\{3B_j\colon j \in J\}$ cover X.*

We also need the following, which we refer to as *Carathéodory's construction.* Essentially, this is a way of obtaining a measure from a function that is defined on a family of subsets of a space.

Definition 3.16. Let $\mathscr{A} \subseteq \mathscr{P}(X)$ and $\phi\colon \mathscr{A} \to \mathbb{R}_{\geq 0}$. We set

$$\Phi^p_\varepsilon(A) = \inf\left\{\sum_i \phi(B_i)^p \colon (B_i)_i \text{ cover } A, \text{radius}(B_i) \leq \varepsilon \text{ for all } i\right\},$$

and define

$$\Phi^p(A) := \limsup_{\varepsilon\to 0} \Phi^p_\varepsilon(A).$$

A proof that this is a measure can be found for example in [Fed69, pp.170].

Note that when $\mathscr{A} = \{B \subseteq X\colon B \text{ is a ball in } X\}$ and $\phi(B) = \text{diam}(B)$, the construction yields the p-dimensional Hausdorff measure.

We need a way of using measurable functions between two spaces to push measures to other measurable spaces.

Definition 3.17. Let $(X,\mu), (Y,\nu)$ be measure spaces and $f\colon X \to Y$ a measurable function. For all measurable sets $A \subseteq Y$ set

$$f_*\mu(A) := \mu(f^{-1}(A)).$$

Then $f_*\mu$ defines a measure on Y that we call the *pushforward measure.*

The next lemma provides a bound of an integral over a curve family. We will be using this lemma with the two measures $\mathscr{H}^p$ and $(f^{-1})_*\mathscr{H}^{p'}$ for a quasiconformal homeomorphism f.

Lemma 3.18. *Let U be an open subset in a metric space X, equipped with two measures μ and ν. Let be a curve family in X, equipped with a measure $d\gamma$. Let $p > 1$. Let us assume that, for every ball B of X that is contained in U,*

$$\int_{\{\gamma\in\Gamma\colon\, \gamma\cap B\neq\emptyset\}} d\gamma \leq \mu\left(\frac{1}{3}B\right)^{1-1/p}.$$

For every ball $B \subseteq U$, we set $\phi(B) = \nu\left(\frac{1}{3}B\right)^{1/p}$. Let Φ^1 be the 1-dimensional measure obtained by Carathéodory's construction from ϕ. Then

$$\int_\Gamma \Phi^1(\gamma) d\gamma \le \nu(U)^{1/p}\mu(U)^{1-1/p}.$$

Proof. We take a covering of U by balls $\frac{1}{3}B_i$ contained in U with diameter less than ε. From Lemma 3.15 we obtain a cover B_i such that the balls $\frac{1}{3}B_i$ do not overlap.

For each B_i we define an indicator function on the family of curves Γ,

$$1_i(\gamma) = \begin{cases} 1 & \text{if } B_i \cap \gamma \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then, because we take the infimum over such sums in the definition of Φ^1, we have

$$\Phi^1_\varepsilon(\gamma) \le \sum_i 1_i(\gamma)\phi(B_i) = \sum_i 1_i(\gamma)\nu(\frac{1}{3}B_i)^{1/p}.$$

When integrating over Φ^1_ε, by Tonelli's theorem we may exchange the summation and integration. Using assumption on μ and applying the Hölder inequaliy, it follows that

$$\begin{aligned}\int_\Gamma \Phi^1_\varepsilon(\gamma) d\gamma &\le \sum_i \left(\int_\Gamma 1_i(\gamma) d\gamma\right) \nu(\frac{1}{3}B_i)^{1/p} \le \sum_i \nu(\frac{1}{3}B_i)^{1/p}\mu(\frac{1}{3}B_i)^{1-1/p} \\ &\le \left(\sum_i \nu(\frac{1}{3}B_i)\right)^{1/p}\left(\sum_i \mu(\frac{1}{3}B_i)\right)^{1-1/p} \le \nu(U)^{1/p}\mu(U)^{1-1/p},\end{aligned}$$

where the last inequality is a consequence of the fact that the balls $\frac{1}{3}B_i$ are pairwise disjoint and contained in U. As $\Phi^1_\varepsilon(\gamma)$ increases when ε decreases to 0, and the bound above holds for every $\varepsilon > 0$, it is still valid in the limit $\varepsilon \to 0$. This concludes the proof. □

We are interested in the case that the metric space is a Carnot group and the two measures are the Hausdorff measure $\mathscr{H}^p$ and the pushforward measure $(f^{-1})_*\mathscr{H}^{p'}$, where f is a quasiconformal homeomorphism. Our goal is to answer the question if $(f^{-1})_*\mathscr{H}^{p'}$ is absolutely continuous with respect to $\mathscr{H}^p$ in order to study the absolute continuity of f. We need the following concept.

Definition 3.19. If f is a homeomorphism between open subsets of Carnot groups with Hausdorff dimension p and p', we define its *Jacobian*

$$f'(x) = \lim_{\varepsilon\to 0} \frac{\mathscr{H}^{p'}(f(B(x,\varepsilon)))}{\mathscr{H}^p(B(x,\varepsilon))}.$$

By [Fed69, pp.152], a sufficient condition for the existence of this limit is $\mathscr{H}^p(2B) \le \text{const.}\mathscr{H}^p(B)$ for every ball B, a condition that is satisfied by (2.2). Thus the limit f' exists and is finite almost everywhere for homeomorphisms between open subsets of Carnot groups.

We can now prove that quasiconformal homeomorphisms map rectifiable curves to rectifiable curves. Given that by Corollary 3.8 it is sufficient to confirm for a basis $\{X_1, \cdots, X_r\}$ of V^1 the differentiability of the curves $s \mapsto f(x \exp(sX_i))$ and by Proposition 3.12 we know that rectifiable curves are differentiable, and given that the curves $s \mapsto \exp(sX_i)$ are rectifiable, this will complete the proof that quasiconformal homeomorphisms are differentiable.

Proposition 3.20. *Let N and N' be Carnot groups, let $U \subseteq N$ and $U' \subseteq N'$ be open with $\mathscr{H}^p(U) < \infty$ and let $f \colon U \to U'$ be a quasiconformal homeomorphism. Then*

(i) N and N' have equal Hausdorff dimension,

(ii) f sends almost every orbit of a left-invariant horizontal vector field to a rectifiable curve,

(iii) the local dilation Lip_f is finite almost everywhere and bounded by $(\mathrm{Lip}_f)^p \leq \eta(1)^p f'$.

Proof. Let U be an open subset of the Carnot group N such that $\mathscr{H}^p(U) < \infty$, where p is the Hausdorff dimension of N . Let $\mu = \mathscr{H}^p$, and let $v \in V^1$ define a horizontal left-invariant vector field. By Ψ we denote the flow of the vector field defined by v. We choose the curve family Γ that consists of curves γ_x with $\gamma_x(t) = \Psi(x, t)$. These are the orbits of v.

Let ω be a biinvariant volume form. This defines a measure $d\gamma$ on the space of orbits that is invariant through left translation and homogeneous with degree $p-1$ under the homotheties $e^{t\alpha}$. Hence, up to a normalisation of ω, one has for every ball B contained in U,

$$\int_{\{\gamma\in\Gamma\colon \gamma\cap B\neq\emptyset\}} d\gamma = \mu\left(\frac{1}{3}B\right)^{1-1/p}. \tag{3.7}$$

It follows that the assumptions of Lemma 3.18 are satisfied.

Let $f \colon U \to U' \subseteq N'$ be a quasiconformal homeomorphism, and let $\nu = (f^{-1})_* \mathscr{H}^{p'}$. Since f is η-quasisymmetric, for every ball B in U there exists a ball B' in N' such that

$$B' \subseteq f\left(\frac{1}{3}B\right) \subseteq f(B) \subseteq \eta(3)B'.$$

By (2.2) there exists some $\sigma > 0$ such that $\mathscr{H}^{p'}(B')^{1/p} \geq \sigma^{p'/p}\mathrm{radius}(B')^{p'/p}$. Hence

$$\phi(B) = \nu\left(\frac{1}{3}B\right)^{1/p} = \left(\mathscr{H}^{p'}\left(f\left(\frac{1}{3}B\right)\right)\right)^{1/p} \geq \mathscr{H}^{p'}(B')^{1/p} \geq \left(\frac{\sigma}{\eta(3)}\right)^{p'/p} \mathrm{radius}(\eta(3)B')^{p'/p},$$

where the first inequality follows from the fact that $B' \subseteq \frac{1}{3}B$. Consider a covering $\{B_i\}_{i\in I}$ of γ. As f is η-quasisymmetric, there exist balls B_i' such that the balls $\eta(3)B_i'$ cover $f(\gamma)$. Further, we have that $f(B) \subseteq \eta(3)B'$, thus it follows that

$$\Phi^1(\gamma) = \limsup_{\varepsilon\to 0} \Phi^1_\varepsilon(\gamma) = \limsup_{\varepsilon\to 0}\left(\inf\left\{\sum_i \phi(B_i)\colon (B_i)_i \text{ cover } \gamma, \text{radius}(B_i) \leq \varepsilon \text{ for all } i\right\}\right)$$

$$\geq \limsup_{\varepsilon\to 0}\left(\inf\left\{\sum_i \left(\frac{\sigma}{\eta(3)}\right)^{p'/p} \text{radius}(\eta(3)B_i')^{p'/p}\colon \right.\right.$$

$$\left.\left.(\eta(3)B_i')_i \text{ cover } f(\gamma), \text{radius}(B_i) \leq \varepsilon \text{ for all } i\right\}\right)$$

$$= \left(\frac{\sigma}{\eta(3)}\right)^{p'/p} \mathscr{H}^{p'/p}(f(\gamma)).$$

By Lemma 3.18, the integral over Φ^1 is bounded and therefore the image of almost every orbit of v under f has a finite p'/p-dimensional Hausdorff measure. This proves (ii).

Further, the Hausdorff dimension of $f(\gamma)$ is at least 1, which implies $p' \geq p$. Repeating this argument for f^{-1} concludes the proof of (i).

To prove (iii), we set $L = \sup\{d(f(x), f(y))\colon d(x,y) < r\}$ and $l = d(f(x), f(\partial B(x,r)))$ for small $r > 0$. When r tends to 0, then $\mathrm{Lip}_f(x) = \limsup \frac{L}{r}$ per definition, and $L/l \leq \eta(1)$, because f maps the 1-annulus (B, B) onto an $\eta(1)$-annulus. This implies that there exists a ball B' such that $B' \subseteq f(B) \subseteq \eta(1)B'$. The radius of B' is thus at most the distance of $f(x)$ and $\partial B(x,r)$, so that $d(f(x), f(y))d(x,y) < r \leq \eta(1)\text{radius}B'$. We conclude that

$$\frac{L}{l} = \frac{\sup\{d(f(x), f(y))\colon d(x,y) < r\}}{d(f(x), f(\partial B(x,r)))}.$$

Thus, for sufficiently small r, it holds that

$$\left(\frac{L}{r}\right)^p \leq \eta(1)^p \left(\frac{l}{r}\right)^p \leq \eta(1)^p \frac{\mathscr{H}^p(f(B(x,r)))}{r^p}.$$

At almost every point x, the right hand side tends to $\eta(1)^p \frac{\mathscr{H}^p(f(B(x,r)))}{\mathscr{H}^p(B(x,r))} = \eta(1)^p f'(x)$ as $r \to 0$. □

Proof (*of Proposition 3.1*). It is clear that the image of rectifiable curves under a Lipschitz map are again rectifiable curves. From Proposition 3.12 we know that the hypotheses of Corollary 3.8 are satisfied, and therefore Lipschitz maps are differentiable.

For quasiconformal homeomorphisms, we have shown in Proposition 3.20 (ii) that the images of rectifiable curves are again rectifiable curves. Again, the statement of the proposition follows from Corollary 3.8 and Proposition 3.12.

In both cases, we obtain the group homomorphism property of the differential from Lemma 3.7, and the fact that, in the case of a map from an open subset of N to N, the differential commutes with $e^{t\alpha}$ from Lemma 3.5. □

4 Absolute continuity of quasiconformal homeomorphisms

4.1 Bijectivity of the differential

In this section, we prove some results regarding absolute continuity of quasiconformal homeomorphisms. These will be needed to show that differentials of quasiconformal homeomorphisms between Carnot groups of Hausdorff dimensions $p > 1$ are group automorphisms. Further, we prove that quasiconformal homeomorphisms are absolutely continuous along almost every curve. This property will be relevant later for some local-to-global arguments.

We recall some general definitions and results.

Definition 4.1. If μ and ν are two measures on the same measurable space $(X, \mathcal{A})$, then μ is said to be *absolutely continuous with respect to* ν if $\mu(A) = 0$ for every set $A \in \mathcal{A}$ for which $\nu(A) = 0$.

Lemma 4.2. [Roy88, Theorem 11.23] *If μ is absolutely continuous with respect to ν, and both measures are σ-finite, then μ has a density. That is, there exists a ν-measurable function f taking values in $[0, \infty)$, such that for any ν-measurable set A, we have*

$$\mu(A) = \int_A f \, d\nu.$$

Definition 4.3. Let μ, ν be measures satisfying the requirements of Lemma 4.2. The density f is also called *Radon-Nikodym derivative* of μ with respect to ν, and denoted $f = \frac{d\mu}{d\nu}$.

Lemma 4.4. [Fed69, Corollary 2.9.20] *A function f between open subsets of Carnot groups is locally absolutely continuous if and only if its distributional derivative is a measure that is absolutely continuous with respect to $\mathscr{H}^p$. If absolute continuity holds then the Radon–Nikodym derivative of μ is equal to the derivative of f almost everywhere.*

We remark that the statement of this lemma usually refers to absolute continuity with respect to the Lebesgue measure, however, in the case of Carnot groups with Carnot-Carathéodory metrics, it can be shown that the Hausdorff measure and the Lebesgue measure, the latter with respect to exponential coordinates, are proportional [LD23, Proposition 8.3.3].

Definition 4.5. Let μ and ν be two measures that are defined on a measurable space (Ω, Σ). We say that μ is *singular with respect to* ν if there exist two disjoint σ-subalgebras $A, B \subseteq \Sigma$ whose union is Σ such that $\mu|_A = 0$ and $\nu|_B = 0$.

Lemma 4.6. [Rud87, Theorem 6.10] *Let λ be a positive σ-finite measure on a σ-algebra Σ in a set X, and let ρ be a measure on Σ. Then, we can write ρ as a sum of two measures,*

$$\rho = \rho_a + \rho_s,$$

where ρ_s is singular with respect to λ and ρ_a is absolutely continuous with respect to λ.

The next lemma gives a bound on the integral over a family of curves similar to Lemma 3.18. This time, however, we obtain a bound in terms of the measure of the set where the Radon-Nikodym derivative is nonzero.

Lemma 4.7. *Let U be an open subset in a metric space X, equipped with two measures μ and ν. Let Γ be a curve family in X, equipped with a measure $d\gamma$. Let $p > 1$. We assume that, for every ball B of X that is contained in U,*

$$\int_{\{\gamma \in \Gamma\colon \gamma \cap B \neq \emptyset\}} d\gamma \leq \mu\left(\frac{1}{3}B\right)^{1-1/p}.$$

For every ball $B \subseteq U$, we set $\phi(B) := \nu\left(\frac{1}{3}B\right)^{1/p}$ and we denote by Φ^1 the 1-dimensional measure obtained by Carathéodory's construction from ϕ. In addition, we assume that there exists a constant ρ such that, for every ball B, we have $\mu(2B) \leq \rho\mu(B)$. Let us define

$$E = \left\{x \in N\colon \frac{d\nu}{d\mu}(x) \neq 0\right\}. \tag{4.1}$$

Then it holds that

$$\int_\Gamma \Phi^1(\gamma) d\gamma \leq \nu(E)^{1/p} \mu(E)^{1-1/p}.$$

Proof. We can assume that $\mu(U) < \infty$. The assumption that $\mu(2B) \leq \rho\mu(B)$ ensures that, at almost every $x \in U$, it is

$$\frac{d\nu}{d\mu}(x) = \lim_{r \to 0} \frac{\nu B(x,r)}{\mu B(x,r)}$$

see for example [Fed69, pp.152]. Let K be a compact subset of $U \setminus E$, and let $\varepsilon > 0$. For every $x \in K$, we choose a ball B_x centred at x with radius at most ε so that $\nu(\frac{1}{3}B_x) \leq \varepsilon^p \mu(\frac{1}{3}B_x)$. If $x \notin K$, we choose B_x centred at x with radius at most ε so that $B_x \cap K = \emptyset$.

The family $\{B_x\colon x \in K \cup K^c\}$ covers $U \setminus E$, and by Lemma 3.15 we can extract from the family $\{B_x : x \in K \cup K^c\}$ of balls a cover of $U \setminus E$ by balls $\{B_i\}_{i\in I}$, where I is a suitable index set such that the balls $\frac{1}{3}B_i$ are pairwise disjoint. Let us denote $J = \{i : B_i \text{ is centred on } K\}$. Then

$$\begin{aligned}
\int_\Gamma \Phi^1_\varepsilon(\gamma)d\gamma &\leq \sum_{i\in J}\phi(B_i)\mu(\frac{1}{3}B_i)^{1-1/p} + \sum_{i\in I\setminus J}\phi(B_i)\mu(\frac{1}{3}B_i)^{1-1/p} \\
&\leq \sum_{i\in J}\varepsilon\mu(\frac{1}{3}B_i) + \sum_{i\in I\setminus J}\phi(B_i)\mu(\frac{1}{3}B_i)^{1-1/p} \\
&\leq \varepsilon\mu(U) + \nu(U\setminus K)^{1/p}\mu(U\setminus K)^{1-1/p}.
\end{aligned}$$

The first inequality follows from the assumption $\int_{\{\gamma\in\Gamma : \gamma\cap B\neq\emptyset\}} d\gamma \leq \mu(\frac{1}{3}B)^{1-1/p}$ together with the observation that, per definition, Φ^1_ε is the infimum over sums of the form $\Phi^1_\varepsilon(\gamma) \leq \sum_i 1_i(\gamma)\phi(B_i)$. The second inequality follows because per choice of ϕ and per choice of the balls B_i we have $\phi(B_i) = \nu(\frac{1}{3}B_x)^{1/p} \leq \varepsilon\mu(\frac{1}{3}B_x)^{1/p}$. The last inequality follows from the fact that the balls $\frac{1}{3}B_i$ are pairwise disjoint, and in the second sum they are in addition disjoint from K.

If we now let ε tend to 0 and K tend to $U \setminus E$, then the term $\varepsilon\mu(U)$ vanishes, whereas the term $\nu(U\setminus K)^{1/p}\mu(U\setminus K)^{1-1/p}$ tends to $\mu(U\setminus(U\setminus E)) = \mu(E)$. We obtain the desired inequality. □

We can use this result to deduce the absolute continuity of the pushforward measure induced by a quasiconformal homeomorphism, and thus, by Lemma 4.4, the absolute continuity of the quasiconformal homeomorphism.

Corollary 4.8. *Let X and X' be metric spaces equipped with measures μ and ν. Let $p > 1$. Assume that there exist positive constants ρ, σ, τ, v such that, if $B \subseteq X$ and $B'(x', r) \subseteq X'$ are two balls, then*

$$\mu(2B) \leq \rho\mu(B) \tag{4.2}$$

and

$$(\sigma r)^p \leq \nu(B'(x', r')) \leq (\tau r)^p. \tag{4.3}$$

Let Γ be a family of curves in X, equipped with a measure $d\gamma$. Assume that for every ball B of X,

$$v\mu(B)^{1-1/p} \leq \int_{\{\gamma\in\Gamma : \gamma\cap B\neq\emptyset\}} d\gamma \leq \mu(B)^{1-1/p}. \tag{4.4}$$

Then for every quasiconformal homeomorphism $f : X \to X'$ the measure $f_*\mu$ is absolutely continuous with respect to ν.

Proof. In the following, we take a covering of a curve γ by balls B_i of radius less than ε. By Lemma 3.15 we can choose this covering such that the balls $\frac{1}{3}B_i$ are disjoint.

Since f is η-quasisymmetric, for every ball B_i the 3-annulus $(\frac{1}{3}B_i, B_i)$ is mapped by f to an $\eta(3)$-annulus. This means that we can find a ball B_i' such that

$$B_i' \subseteq f(\frac{1}{3}B_i) \subseteq f(\frac{1}{3}B_i) \subseteq \eta(3)B_i'.$$

Then, setting $\phi(B) = \nu(\frac{1}{3}B)$, we have

$$\begin{aligned}\Phi^1(\gamma) &= \limsup_{\varepsilon\to 0}\Phi^1_\varepsilon(\gamma) = \limsup_{\varepsilon\to 0}\sum_i \phi(B_i) = \limsup_{\varepsilon\to 0}\sum_i \nu\left(\frac{1}{3}B_i\right)^{1/p} \\ &\geq \limsup_{\varepsilon\to 0}\sum_i \nu\left(B_i'\right)^{1/p} \geq \limsup_{\varepsilon\to 0}\sum_i \sigma\varepsilon \geq \frac{\sigma}{\eta(3)}\mathscr{H}^1 f(\gamma).\end{aligned}$$

The first inequality follows from the subset relation established above and from the pairwise disjointness of the balls $\frac{1}{3}B_i$, the second inequality follows from the assumption (4.3), and the third from the definition of the Hausdorff measure.

Let $x \in X$ and let $r > 0$. As f^{-1} is η-quasisymmetric, and $B'(f(x), r), B'(f(x), 2r)$ is a 2-annulus which is taken to an $\eta(2)$-annulus by f^{-1}, there exists a ball B such that

$$\eta(2)^{-1}B \subseteq f^{-1}B'(f(x), r) \subseteq f^{-1}B'(f(x), 2r) \subseteq B.$$

We may choose B centred at x. If γ intersects with $\eta(2)^{-1}B$ but is not contained in B, then $f(\gamma)$ joins some point in $B'(f(x), r)$ to some point in the complement of $B'(f(x), 2r)$. Its length is at least r since it intersects both boundaries of $B'(f(x), 2r) \setminus B'(f(x), r)$, and by the inequality derived above and the observation that $\mathscr{H}^1(f(\gamma)) \geq \text{length} f(\gamma)$, it follows that

$$\Phi^1(\gamma) \geq \frac{\sigma}{\eta(3)}r. \tag{4.5}$$

We apply Lemma 4.7 to the set B equipped with the measures μ and $(f^{-1})_*\nu$ and the curve family

$$\Gamma(B) = \left\{\gamma \in \Gamma \colon \gamma \cap \eta(2)^{-1}B \neq \emptyset\right\}.$$

Note that the prerequisites are satisfied by the assumptions (4.2) and (4.4). Applying (4.5) and Lemma 4.7, it follows that

$$r\frac{\sigma}{\eta(3)}\int_{\Gamma(B)} d\gamma \leq \int_{\Gamma(B)} \Phi^1(\gamma)d\gamma \leq \rho^2\nu(f(B))^{1/p}\mu(E \cap B)^{1-1/p}.$$

Since f is η-quasisymmetric and $(\eta(2)^{-1}B, B)$ is an $\eta(2)$-annulus which is sent by f to an $(\eta\circ\eta)$-

annulus, there exists a ball $B' \subseteq X'$ such that

$$B' \subseteq f(\frac{1}{\eta(2)}B) \subseteq f(B) \subseteq (\eta \circ \eta)(2)B'.$$

The last subset relation together with the assumption (4.3) yields

$$\nu(f(B))^{1/p} \leq \nu((\eta \circ \eta)(2)B')^{1/p} \leq \tau(\eta \circ \eta)(2)r,$$

because $\eta(2)^{-1}B \subseteq f^{-1}B'(f(x), r)$. Thus $f(\eta(2)^{-1}B) \subseteq B'(f(x), r)$, and $B' \subseteq f(\frac{1}{\eta(2)}B)$, so that $B' \subseteq B(f(x), r)$.

The assumption (4.4) and the choice of $\Gamma(B)$ such that all curves intersect B imply

$$\int_{\Gamma(B)} d\gamma \geq v\mu(B)^{1-1/p}.$$

Combining these last three inequalities, one gets, with E as in (4.1), the bound

$$\frac{\mu(E \cap B)}{\mu(B)} \geq \left(\frac{\sigma v}{\rho^2 \tau(\eta \circ \eta)(2)\eta(3)}\right)^{(1-1/p)^{-1}}, \tag{4.6}$$

where the lower bound only depends on the constants in the statement and the function η.

If $p > 1$, then $1 - \frac{1}{p} > 0$, and the inequality (4.6) gives $\frac{\mu(E \cap B)}{\mu(B)} \geq \text{const} > 0$. It follows that x, the centre of the ball B, cannot be a μ-density point of $X \setminus E$.

We can deduce from this that the Jacobian f' of f is almost everywhere nonzero. For every measurable subset $A \subseteq X$, we have the inequality (see [Fed69, Theorem 2.9.7])

$$\nu(f(A)) \geq \int_A f' d\mu.$$

If $\nu(f(A)) = 0$ then f' vanishes almost everywhere on A. However, we have established that f' is almost everywhere nonzero, so that $\mu(A) = 0$ follows. With Lemma 4.4, we conclude that f is absolutely continuous. $\square$

The following result states one of the central properties of quasiconformal homeomorphisms between Carnot groups with regard to the proof of Theorem 8.1, which is that their differentials are automorphisms of Carnot groups.

Proposition 4.9. *Let N and N' be Carnot groups with Hausdorff dimensions p and p', where $p, p' > 1$, let $U \subseteq N$ and $U' \subseteq N'$ be open with $\mathscr{H}^p(U) < \infty$ and let $f\colon U \to U'$ be a quasiconformal homeomorphism. Then, f is absolutely continuous, and its differential is a group isomorphism almost everywhere.*

Proof. We take X and X' to be the open subsets of Carnot groups equipped with the respective Carnot-Carathéodory metrics, and Γ to be the set of orbits of left-invariant vector fields. We equip X with the measures $\mu = \mathscr{H}^p$ and $\nu = (f^{-1})_* \mathscr{H}^{p'}$. The assumptions of Corollary 4.8 are satisfied, as we saw in the proof of Proposition 3.20.

By Proposition 3.1, the differential $Df(x)$ exists at almost every point x and by Corollary 4.8 the Jacobian $f(x)$ is nonzero. Let x be such a point. Set

$$f_t(x, \mu) = e^{t\alpha'} \left(f(x)^{-1} f(xe^{-t\alpha}\mu) \right).$$

Again by Proposition 3.1, the maps $f_t(x, \cdot)$ converge uniformly to $Df(x)$. Consequently, if B denotes the unit ball in the group, we have

$$\bigcap_{T\to\infty} \bigcup_{t>T} f_t(x, B) \subseteq Df(x)(B).$$

Thus, if $Df(x)$ is not surjective, then

$$0 = \mathscr{H}^p Df(x)(B) \geq \lim_{t\to\infty} \mathscr{H}^p f_t(x, B) = \lim_{t\to\infty} e^{tp} \mathscr{H}^p f(B(1, e^{-t})) = f'(x),$$

but from Corollary 4.8 we know that these points form a set of zero measure.

We claim that the surjectivity of $Df(x)$ implies that it is also injective. Recall that the exponential map of a Carnot group is a diffeomorphism, so that we can associate with $Df(x)$ a globally defined map g on $\mathfrak{n} = \mathrm{Lie}(N)$. Clearly, if $Df(x)$ is surjective, so is g. But g is a linear map on a finite-dimensional vector space, and therefore injective. The injectivity of $Df(x)$ follows. □

4.2 Absolute continuity on lines

The next step in the study of the regularity of quasiconformal transformations is the so-called *ACL property*, which describes absolute continuity of quasiconformal homeomorphisms along almost every line. We first introduce some terminology.

Definition 4.10. Let X be a metric space with distance function d. A curve $\gamma\colon I \to X$ is a *line* if it realises the distance between any two of its points.

Definition 4.11. Let X be a metric space. A subset $a \subseteq X$ is a *k-ball* if (a, a) is a k-annulus, that is, if there exists a ball B such that $B \subseteq a \subseteq kB$.

Definition 4.12. Let ϕ be a function taking positive values on the power set of X. For $l > 1$, we define a new function on the power set of X by setting

$$\tilde{\phi}(a) = \sup\{\phi(\tilde{a}) : (a, \tilde{a}) \text{ is a } l\text{-annulus}\}.$$

Let $p > 0$. We denote by $\Phi^{p,k}$, respectively $\tilde{\Phi}^{p,k}$, the measure obtained by Carathéodory's construction by summing ϕ, respectively $\tilde{\phi}$, on the k-balls.

Let Γ be a curve family in X, and let $p \geq 0, k \geq 1$ and $l \geq k$ be real numbers. We define the *coarse modulus* of Γ as the collection of numbers $M^{p,k,l,m}(\Gamma) = \inf \tilde{\phi}_l^{p,k}$, where the infimum is taken over all functions ϕ such that for every $\gamma \in \Gamma$, we have $\Phi^{1,m}(\gamma) > 1$.

Proposition 4.13. [Pan89a] *The following hold.*

- *The notion of "almost every curve" is preserved by quasiconformal homeomorphisms.*
- *In a Carnot group, let Γ be a family of orbits of a horizontal vector field $v \in V^1$. If $\int_\Gamma d\gamma > 0$, then $M^{p,k,l,m}(\Gamma) > 0$ for every $k \geq 1, l > 4k$ and $m \geq 1$. In particular, if a property is satisfied by almost every curve, then it is satisfied by almost every orbit of v.*

The following lemma allows to derive absolute continuity of quasiconformal homeomorphisms on lines, which we later use to deduce global statements from local properties.

Lemma 4.14. *Let X and X' be metric spaces equipped with measures μ and ν. Let us assume that there exist constants σ and τ such that, for every ball $B \subseteq X$ and $B' \subseteq X'$, $\mu(B) \leq (\tau \mathrm{diam}(B))^{1/p}$, and $(\sigma \mathrm{diam}(B'))^{1/p} \leq \nu(B')$. Let $f\colon X \to X'$ be a η-quasiconformal homeomorphism such that $(f^{-1})_*\nu$ is absolutely continuous with respect to μ. Then, the restriction of f to almost every rectifiable curve is absolutely continuous with respect to 1-dimensional Hausdorff measures.*

Proof. Without loss of generality, one can assume $\nu(X') < \infty$. Let γ be a rectifiable curve in X. We define a positive measure by $\rho(E) = \mathscr{H}^1(f(E))$ on γ. By Lemma 4.6, we can decompose ρ as $\rho = u\mathscr{H}^1_{|\gamma|} + \rho_s$, where ρ_s is singular with respect to $\mathscr{H}^1$ and $u\mathscr{H}^1_{|\gamma|}$ is absolutely continuous with respect to ρ. We set

$$\Gamma_n = \left\{\gamma \text{ curve in } X\colon \rho_s(\gamma) \geq \frac{1}{n}\right\}.$$

If $\rho_s(\gamma) \neq 0$ then $\rho(\gamma) \neq 0$, but as ρ_s is singular with respect to $\mathscr{H}^1$, we know that not all $\mathscr{H}^1$-zero-sets are ρ-zero-sets. It follows that the union of the families Γ_n is the family of curves along which f is not absolutely continuous, and we prove that, for every $l, m \geq 1$, we have $M^{p,1,l,m}(\Gamma_n) = 0$.

Let $R > 0$ and set $\chi(t) = \max\{1, \frac{t}{R}\}$. We define a function ϕ on the subsets a of X by setting

$$\phi(a) = \chi\left(\frac{\mathrm{diam} f(a)}{\mathrm{diam}(a)}\right) \mathrm{diam} f(a),$$

where $\mathrm{diam}(A)$ denotes the diameter of the set A.

We claim that $t \leq R + t\chi(t)$ for all $t > 0$. To see that this is true, first note that if $\chi(t) = 1$, then $t \leq R$ and $t \leq R + t = R + t\chi(t)$ clearly holds true. If $\chi(t) = \frac{t}{R} > 1$, then $t > R$ and as $\frac{t}{R} \leq 1 + \frac{t^2}{R^2}$ holds for $\frac{t}{R} > 1$, the inequality $t \leq R + t\chi(t)$ follows.

We insert $t = \mathrm{diam}(f(a))/\mathrm{diam}(a)$, and then multiply by $\mathrm{diam}(a)$ to get

$$\mathrm{diam} f(a) \leq R\mathrm{diam}(a) + \phi(a). \tag{4.7}$$

On the other hand, for all $0 < s < t$, it is $\frac{\chi(t)}{\chi(s)} \leq \frac{t}{s}$. Note that this is equivalent to demanding that either $t, s \geq R$ or $t, s \leq R$. We then have $\tilde{\phi}(a) = \sup\{\phi(\tilde{a}) : (a, \tilde{a}) \text{ is a } l\text{-annulus}\}$, thus

$$\tilde{\phi}_l(a) \leq \eta(l)^2 \phi(a). \tag{4.8}$$

Let us now prove that for every curve γ in X and for all intervals E of γ, we have

$$\mathscr{H}^1(f(E)) \leq \mathrm{const.}(R\mathscr{H}^1(E) + \Phi^{1,m}(E)).$$

Take a cover $\{a_i\}_{i \in I}$ of E by m-balls with diameter at most ε such that

$$\sum_i \phi(a_i) \leq \Phi^{1,m}(E) + \varepsilon. \tag{4.9}$$

Since the a_i are m-balls, by definition there are balls B_i such that

$$\frac{1}{m} B_i \subseteq a_i \subseteq B_i.$$

We take the subfamily $\{B_i\}_{i \in J}$ that only consists of the balls that intersect E. By Lemma 3.15, we can extract from the covering $\{3B_i\}_{i \in J}$ a subcovering with disjoint balls such that the balls $9B_i$ cover E. By (4.8), we have

$$\tilde{\phi}_l(a) = \sup\{\phi(\tilde{a}) : (a, \tilde{a}) \text{ is a } l\text{-annulus}\} \leq \eta(l)^2 \phi(a),$$

and since $(a_i, 9B_i)$ is a $9m$-annulus, we can find a ball B such that $B \subseteq a_i \subseteq 9B_i \subseteq 9mB$, by choosing $B = \frac{1}{m}B_i$ from before, it is

$$\phi(9B_i) \leq \eta(9m)^2\phi(a_i). \tag{4.10}$$

When $\varepsilon \geq \operatorname{diam}(a_i)$ is small enough, E is not contained in any of the balls $3B_i$. But as E intersects B_i, we have $\operatorname{diam}(B_i) \leq \mathscr{H}^1(E \cap 3B_i)$, which implies

$$\sum_i \operatorname{diam}(B_i) \leq \mathscr{H}^1(E), \tag{4.11}$$

since the balls $3B_i$ are disjoint. Hence, for a uniformly continuous function $\varepsilon'(\varepsilon)$, it is

$$\begin{aligned}\mathscr{H}^1_{\varepsilon'}(f(E)) &\leq \sum_i \operatorname{diam}(f(9B_i)) \leq \sum_i R\operatorname{diam}(9B_i) + \phi(9B_i)\\ &\leq 9R\mathscr{H}^1(E) + \eta(9m)^2\Phi^{1,m}(E) + \eta(9m)^2\varepsilon,\end{aligned}$$

this follows from (4.7), (4.9), (4.10) and (4.11). As ρ_s is singular with respect to $\mathscr{H}^1$, this proves that for every rectifiable curve γ, we have $\rho_s(\gamma) \leq \eta(9m)^2\Phi^{1,m}(\gamma)$. Consequently, for every p, k, l, m, n, the coarse modulus is given by

$$M^{p,k,l,m}(\Gamma_n) \leq n^p\eta(9m)^{2p}\tilde{\phi}^{p,1}_l(X).$$

With (4.8), it is sufficient to prove that $\Phi^{p,1}(X)$ tends to 0 when R tends to ∞.

For every $x \in X$, we set

$$\Theta(x) := \limsup_{B\to x}\frac{\nu f(B)}{\mu(B)}.$$

Let $A(t) = \{x \in X : \Theta(x) \geq t^p\}$ and $\omega(t) = \nu f(A(t))$. If we consider the limit $t \to \infty$, we see that $\Theta(x) \geq t^p$ implies $\mu(B) \to 0$ because we assumed $\nu(X') < \infty$. But since f is absolutely continuous, μ-zero measure sets are sent to ν-zero measure sets, so that $\omega(t)$ tends to 0 when t tends to ∞.

We fix $\varepsilon > 0$ and for every $x \notin A(t)$, we choose a ball B_x centred at x with a radius less than ε such that

$$\nu f(3B_x) \leq t^p\mu(3B_x).$$

As f is quasisymmetric, there is a ball B' in X' such that

$$B' \subseteq f(3B_x) \subseteq \eta(1)B'.$$

We then have

$$\phi(3B_x) \leq C\frac{t^2}{R}f(3B_x)^{1/p},$$

where $C = C(\eta, \sigma, \tau)$ only depends on the constants in the statement. Then

$$\frac{\sigma}{\eta(1)}\mathrm{diam} f(3B_x) \leq \sigma \mathrm{diam}(B') \leq \nu(B')^{1/p} \leq t\mu(3B_x)^{1/p} \leq t\tau \mathrm{diam}(3B_x),$$

where the last inequality is one of our assumptions. Considering that $\chi(t) \leq \frac{t}{R}$, we can use parts of both sides of the previous sequence of inequalities to see that

$$\phi(3B_x) = \chi\left(\frac{\mathrm{diam}(f(3B_x))}{\mathrm{diam}(3B_x)}\right)\mathrm{diam}(f(3B_x)) \leq \frac{(\mathrm{diam}(f(3B_x)))^2}{\mathrm{diam}(3B_x)} \leq \frac{3\tau\eta(1)^2t^2}{\sigma^2 R}\nu f(B_x)^{1/p}.$$

Again by Lemma 3.15 we extract a subfamily $\{3B_i\}$ which covers $X \setminus A(t)$ such that the B_i are disjoint, and we obtain

$$\Phi_\varepsilon^{p,1}(X \setminus A(t)) \leq \sum_x \phi(3B_x)^p \leq \left(\frac{Ct^2}{R}\right)^p \sum_i \nu f(B_i) \leq \left(\frac{Ct^2}{R}\right)^p \nu f(X).$$

As $\chi(t) \leq 1$, we have $\sigma\phi(B) \leq \eta(1)\nu f(B)^{1/p}$. Covering $f(A(t))$ with a set of balls $\{B\}$, this yields

$$\Phi^{p,1}(A(t)) \leq \left(\frac{\eta(1)}{\sigma}\right)^p \nu f(A(t)).$$

By adding the bounds for $\Phi^{p,1}(A(t))$ and $\Phi_\varepsilon^{p,1}(X \setminus A(t))$, and considering that the measure $\Phi^{p,1}$ is subadditive, it follows that

$$\Phi^{p,1}(X) \leq \inf_{t>0}\left\{\left(\frac{Ct^2}{R}\right)^p \nu f(X) + \frac{\eta(1)}{\sigma}\omega(t)\right\}.$$

The first summand clearly tends to 0 as $R \to \infty$. Further, as $\omega(t) \to 0$ as t tends to ∞, it follows that $\Phi^{p,1}(X)$ tends to 0 as R tends to ∞. This concludes the proof. □

In Corollary 4.8 we saw that quasiconformal homeomorphisms between open subsets of Carnot groups are absolutely continuous. As lines are rectifiable curves, Lemma 4.14 allows us to conclude that the restrictions of quasiconformal homeomorphisms to almost every rectifiable curve are absolutely continuous with respect to 1-dimensional Hausdorff measures.

Proposition 4.15. *A quasiconformal homeomorphism between open subsets of Carnot groups with Hausdorff dimension $p > 1$ is absolutely continuous on almost every line.*

The absolute continuity on lines allows to consider another invariant besides the coarse modulus, namely the capacity.

Definition 4.16. Let U be an open subset in a Carnot group N. The space $ACL^p(U)$ is the space of continuous functions u on U which are absolutely continuous on almost every line, and whose local dilation Lip_u is in $L^p(U)$.

A *capacitor* is an open subset $C \subseteq U$ together with two subsets $\partial_0 C$ and $\partial_1 C$ of C, called plates. The *p-capacity* of $(C, \partial_0 C, \partial_1 C)$ is

$$\inf \left\{ \int_C \mathrm{Lip}_u^p d\mathscr{H}^p \colon u \in ACL^p(C) \cap C^0(C), u = 0 \text{ on } \partial_0 C \text{ and } u = 1 \text{ on } \partial_1 C \right\}.$$

Remark 4.17. Consider for $0 < r < R$ the spherical capacitor

$$C = B(x, R) \setminus B(x, r), \quad \partial_0 C = \partial B(x, r), \quad \partial_0 C = \partial B(x, r).$$

It clearly has a nonzero capacity, which, by invariance of the capacity under dilations, only depends on the ratio R/r. We denote the capacity of such a spherical capacitor $\varphi(R/r)$.

Remark 4.18. For our purposes, the exact values of φ is irrelevant, and we only need the property that φ is increasing as its argument increases.

We are interested in quasiconformal homeomorphisms whose differential has a special form, namely the product of an isometry and a homothety. They are called 1-quasiconformal homeomorphisms, and we can show that 1-quasiconformal homeomorphisms preserve capacities.

Definition 4.19. A map f between Carnot groups N and N' is a *similarity* if f is the product of a homothety $e^{t\alpha}$ and a linear isometry.

Definition 4.20. A homeomorphism between open sets of Carnot groups is *1-quasiconformal* if it is η-quasiconformal for some function η and if its differential is a similarity almost everywhere.

Lemma 4.21. *A 1-quasiconformal homeomorphism $f \colon U \to V$ between open subsets of a Carnot group with Hausdorff dimension p induces an isometry from $ACL^p(U)$ to $ACL^p(V)$. In particular, it preserves capacities.*

Proof. By Proposition 4.13 and Proposition 4.15, quasiconformal homeomorphisms map functions that are absolutely continuous along almost every line to functions that are absolutely continuous along almost every line. If $Df(x)$ is a similarity, we have $f'(x) = \mathrm{Lip}_f(x)^p$, so that for every locally Lipschitz function v on V,

$$\mathrm{Lip}_{v \circ f}(x)^p \leq \mathrm{Lip}_v(f(x))^p \mathrm{Lip}_f(x)^p = \mathrm{Lip}_v(f(x))^p f'(x).$$

We conclude that

$$\left(\left\|\mathrm{Lip}_{v\circ f}\right\|_p\right)^p = \int \mathrm{Lip}_{v\circ f}^p d\mathscr{H}^p \leq \int \mathrm{Lip}_v(f(x))^p f'(x) d\mathscr{H}^p = \left(\|\mathrm{Lip}_v\|_p\right)^p,$$

which proves that 1-quasiconformal homeomorphisms preserve the spaces ACL^p. □

5 Quasiisometries of rank-one symmetric spaces

5.1 The rank-one symmetric spaces

This section introduces the rank-one symmetric spaces and discusses quasiisometries on them. In Section 5.1, we recall a few facts about symmetric spaces and present models for the irreducible rank-one symmetric spaces of non-compact type, which we also refer to as hyperbolic spaces. We show that the boundaries of the rank-one symmetric spaces can be equipped with a Carnot group structure. This is done in Section 5.2. Finally, in Section 5.3, we show that quasiisometries can be extended to the boundaries and prove that those extensions are quasiconformal homeomorphisms.

Symmetric spaces are a class of Riemannian manifolds, each point of which is the fixed point of an involutive isometry. Their isometry groups are Lie groups acting transitively on the manifolds. Choosing any base point p in a symmetric space X and by K_p denoting the subgroup of the isometry group $\mathrm{Isom}(X)$ fixing p, we can identify $X \cong \mathrm{Isom}_\circ(X)/K_p$. This allows for a powerful combination of the tools of Riemannian geometry and Lie theory to study these spaces.

Every symmetric space can be decomposed into a product of *irreducible symmetric spaces*, which cannot be written as a product space of two symmetric spaces. Each irreducible symmetric space is of a well-defined *type*, either compact, non-compact or Euclidean, based on its sectional curvature being non-negative, non-positive or identically zero respectively. The *rank* of a symmetric space is the maximal dimension of a flat, totally geodesic submanifold [Hel78].

There are three families of irreducible rank-one symmetric spaces of non-compact type and one exceptional case. The three families are the real hyperbolic spaces that we denote $\mathbb{R}\mathbf{H}^n$, the complex and quaternionic variants $\mathbb{C}\mathbf{H}^n$ and $\mathbb{H}\mathbf{H}^n$, where for all three families we have $n \geq 2$, and the exceptional case is the octonionic hyperbolic plane $\mathbb{O}\mathbf{H}^2$. The real hyperbolic spaces have constant sectional curvature -1, whereas the other hyperbolic spaces have sectional curvature between -4 and -1.

Convention. We agree that by *rank-one symmetric space* we always refer to an irreducible rank one symmetric space of non-compact type. We will also refer to the rank-one symmetric spaces as *hyperbolic spaces.*

Notation. For general statements, we will use the notation $\mathbb{K}\mathbf{H}^n$ to denote an arbitrary hyperbolic space, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $n \geq 2$. Whenever $\mathbb{K} = \mathbb{O}$, we take $n = 2$.

As quotients of the connected component containing the identity of their isometry groups by a point stabiliser, we may identify the rank-one symmetric spaces with

$$\begin{aligned}
\mathbb{R}\mathbf{H}^n &\cong SO_o(n,1)\big/SO(n), \\
\mathbb{C}\mathbf{H}^n &\cong SU(n,1)\big/S(U(n) \times U(1)), \\
\mathbb{H}\mathbf{H}^n &\cong Sp(n,1)\big/Sp(n) \times Sp(1), \\
\mathbb{O}\mathbf{H}^2 &\cong F_4\big/\mathrm{Spin}(9),
\end{aligned}$$

see for example [Hel78, pp.452]. Due to their non-positive sectional curvature, we can find global parameterisations of the hyperbolic spaces by the Cartan-Hadamard theorem, and it is useful to choose parameterisations that are derived from these Lie group structures. This leads to the following model for the three families of hyperbolic spaces (see also [LD23, Section 10]). The case of the octonionic hyperbolic plane cannot be treated in the same way. We only provide a brief sketch of a way to model $\mathbb{O}\mathbf{H}^2$ and refer the reader to [Mos73, Section 19] for a comprehensive treatment.

From now on we sometimes work with real, complex and quaternionic vectors and matrices explicitly. We use the following conventions for our notation, which we state explicitly only for the quaternions. Our notation for the other cases is analogous and can be viewed as a special case of the notation introduced below.

Notation. A quaternion u can be written as $u = a + ib + jc + kd$, where $a, b, c, d \in \mathbb{R}$. We define

- the *conjugate* $\overline{u} = a - ib - jc - kd$,
- the *absolute value* $|u| = \sqrt{u\overline{u}}$,
- the *real part* $\mathrm{Re}\ u = \frac{u+\overline{u}}{2}$,
- the *imaginary part* $\mathrm{Im}\ u = \frac{u-\overline{u}}{2}$.

Further, if A is a vector or matrix with quaternionic entries, we define its *Hermitian transpose* A^* as the matrix that is obtained by transposing A and conjugating every entry.

Definition 5.1. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $n \geq 2$. We model the hyperbolic space $\mathbb{K}\mathbf{H}^n$ as the semidirect product

$$\mathbb{K}\mathbf{H}^n := \left(\mathbb{K}^{n-1} \ltimes \text{Im } \mathbb{K}\right) \rtimes \mathbb{R}_+ \tag{5.1}$$

with the multiplication law

$$(v, s, a)(w, t, b) = (v + aw, s + at + \text{Im } (aw^* v), ab),$$

and equip it with the left-invariant Riemannian metric g given by

$$g_{(0,0,1)}((v, s, a), (v, s, a)) = a^2 + \frac{v^* v}{2} - \frac{s^2}{4}, \tag{5.2}$$

where $(v, s, a) \in T_{(0,0,1)}\mathbb{K}\mathbf{H}^n \cong \mathbb{K}^{n-1} \times \text{Im } \mathbb{K} \times \mathbb{R}$.

Note that the square of any nonzero $s \in \text{Im } \mathbb{K}$ is a negative real number, so that the inner product $g_{(0,0,1)}$ is indeed positive definite.

The octonionic hyperbolic plane is different than the others because multiplication of octonions is not associative, so that we cannot define octonionic vector spaces. To deal with this, one uses the fact that subalgebras of $\mathbb{O}$ with two generators are associative to define an associative $\mathbb{O}$-algebra whose automorphism group is F_4. The equivalence relation on the set of triplets of octonions which lie in one associative subalgebra defined by $z \sim w$ if there exists some λ in an associative subalgebra of $\mathbb{O}$ containing the entries of z such that $w = z\lambda$ yields a quotient space that can be thought of as a projective octonionic plane, and a subset of it is identified with $\mathbb{O}\mathbf{H}^2$. Its point stabilisers are, up to conjugation, isomorphic to Spin(9) [Mos73].

To illustrate our further considerations, it is often useful to have an explicit model, such as (5.1), of the hyperbolic spaces at hand. Dealing with the case of $\mathbb{O}\mathbf{H}^2$ separately every time would go beyond the scope of this thesis, so that we sometimes omit an explicit treatment of it. However, all arguments that are essential for the proof of Theorem 8.1 are valid in full generality and apply to $\mathbb{O}\mathbf{H}^2$ as well as to the other cases.

5.2 Boundaries at infinity and their Carnot group structure

One can attach to the hyperbolic spaces a sphere at infinity, and we will see that it naturally carries a Carnot group structure. To define this formally, we introduce the following notion.

Definition 5.2. Two geodesics c and c' are *asymptotic* if there exists a constant K such that $d(c(t), c'(t)) \leq K$ for all $t \geq 0$.

We use this to define an equivalence relation $\sim$ on the set of geodesics:

$$c \sim c' \quad :\Leftrightarrow \quad c, c' \text{ are asymptotic.}$$

It is not difficult to show that $\sim$ defines an equivalence relation, see for example [EO73, p.48]. We are now equipped to define the boundary at infinity of the hyperbolic spaces.

Definition 5.3. We define the *boundary at infinity* of $\mathbb{K}\mathbf{H}^n$, denoted $\partial\mathbb{K}\mathbf{H}^n$, as the set of equivalence classes of geodesics for the equivalence relation $\sim$.

We sometimes omit the words "at infinity" and just speak of the boundary of a hyperbolic space. An explicit description of the boundaries can be obtained, and we briefly sketch a possible way of doing so, following [BH99, Chapter II.10]. We start by introducing horospheres in $\mathbb{K}\mathbf{H}^n$. These are the limits of sequences of unboundedly large spheres which are all tangent to one fixed hyperplane.

Definition 5.4. The spheres centred at $y \in \mathbb{K}\mathbf{H}^n$ are the level sets of the functions

$$\rho_y(x) = d(x, y).$$

To construct a sequence of increasingly large spheres with a common tangent hyperplane starting with some sphere $\rho_y^{-1}(r)$, we fix a geodesic c that passes though the centre of the ball $\rho_y^{-1}(r)$ at time $t = 0$. This geodesic will be orthogonal to the common tangent hyperplane.

Let x be one of the two points where the geodesic meets the sphere $\rho_y^{-1}(r)$, for example we choose $x \in c|_{(-\infty,0]} \cap \rho_y^{-1}(r)$. Let $H_{x,r}$ be the hyperplane containing x that is tangent to $\rho_y^{-1}(r)$. This will be the common tangent hyperplane.

For $t \geq 0$ we set $r_t \in [0, \infty)$ such that $\rho_{c(t)}^{-1}(r_t)$ is tangent to $H_{x,r}$. As $t \to \infty$, the sequence of balls $\rho_{c(t)}^{-1}(r_t)$ converges and we call its limit the *horosphere centred at* $[c]$,

$$H_{r,[c]} = \lim_{t\to\infty} \rho_{c(t)}^{-1}(r_t).$$

We fix a point $o \in \mathbb{K}\mathbf{H}^n$, and a geodesic c such that $c(0) = o$. Let A denote the one-parameter group of transvections along c, that is, $A = \{A(t) \colon t \in \mathbb{R}\}$, where $A(t).c(t') = c(t + t')$, and let N be the subgroup of $\operatorname{Isom}(\mathbb{K}\mathbf{H}^n)$ that acts simply transitively on the horosphere centred at $[c]$ containing o. The hyperbolic space can then be identified with $N \rtimes A$. Up to conjugation, which geometrically can be thought of as moving the base point o and choosing another geodesic c, these are exactly the groups from (5.1), that is, A is the abelian group $(\mathbb{R}_+, \cdot)$, and $N = \mathbb{K}^{n-1} \ltimes \mathrm{Im}\ \mathbb{K}$, with multiplication law

$$(v, s)(w, t) = (v + w, s + t + \mathrm{Im}\ (w^* v)).$$

We remark that it also holds for the octonionic hyperbolic plane that there is a nilpotent Lie group N and an abelian Lie group A with $\mathbb{O}\mathbf{H}^2 \cong N \rtimes A$. This is evident from the Iwasawa decomposition [Hel78, p.403].

We proceed by defining a map from N to $\partial\mathbb{K}\mathbf{H}^n \setminus \{[c]\}$, by setting $c^- : \mathbb{R} \to \mathbb{K}\mathbf{H}^n, t \mapsto c(-t)$, and $n \mapsto [n.c^-]$. It can be shown that this map is a homeomorphism from N to $\partial\mathbb{K}\mathbf{H}^n \setminus \{[c]\}$, and we conclude that the boundaries at infinity of the hyperbolic spaces, after removing one point, are identified with the groups N. Usually, the point $[c] \in \partial\mathbb{K}\mathbf{H}^n$ is called ∞, and we adopt this notation from now on.

In the following, we show that N is a Carnot group and describe its subRiemannian structure. If $\mathbb{K} = \mathbb{R}$, then the group N is abelian. In the other cases, the Lie algebra $\mathfrak{n}$ admits a gradation $\mathfrak{n} = V^1 \oplus V^2$, where we identify V^1 with $\mathbb{K}^{n-1}$ and V^2 with Im $\mathbb{K}$. The Lie bracket can be expressed as

$$[x, y] = \mathrm{Im}\ (y^* x) \qquad \text{for } x, y \in V^1, \tag{5.3}$$

which shows that $[V^1, V^1] = V^2$, and V^2 is the centre of $\mathfrak{n}$ that we denote $\mathcal{Z}(\mathfrak{n})$. This defines a derivation α on $\mathfrak{n}$ such that $V^1 = \ker(\alpha - 1)$ by setting $\alpha|_{V^1} = id_{V^1}$ and $\alpha|_{V^2} = 2id_{V^2}$, so that N with the derivation α is a Carnot group. Further, we obtain a distribution Δ on N by setting

$$\Delta_p = d_e L_p(V^1) \qquad \text{for } p \in N. \tag{5.4}$$

Following Section 2.2, by choosing an inner product on V^1, we obtain a Carnot-Carathéodory metric on N. This is done explicitly below.

It is worth noting that α can also be given a geometric meaning. Indeed, the transvection $A(1)$ along c is precisely the Lie group equivalent of α, and more generally we have

$$A(t) = e^{t\alpha}.$$

In the following we use the identification of $\mathbb{K}\mathbf{H}^n$ with $N \rtimes A$ to denote points in $\mathbb{K}\mathbf{H}^n$ by

$(n,t) \in N \rtimes A$. Given our choice of a base point in the hyperbolic space, the point o corresponds to $(0,1)$, which is the identity element in N. The metric g of $\mathbb{K}\mathbf{H}^n$ that was defined in (5.2) can be written as

$$g_{(n,t)} = g^N_{(n,t)} \oplus dt^2, \tag{5.5}$$

where the left-invariant metrics $g^N_{(n,t)}$ on N have matrices of the form

$$g^N_{(0,t)} = \begin{pmatrix} e^{2t} I_{(n-1)\dim_{\mathbb{R}} \mathbb{K}} & 0 \\ 0 & e^{4t} I_{\dim_{\mathbb{R}} \operatorname{Im} \mathbb{K}} \end{pmatrix} \tag{5.6}$$

in the gradation $V^1 \oplus V^2$. This naturally (but not canonically!) provides us with a left-invariant metric on the boundary at infinity by restricting the Riemannian metric on $\mathbb{K}\mathbf{H}^n$ to the horosphere centred at ∞ containing $o \in \mathbb{K}\mathbf{H}^n$. Since the horospheres are embedded submanifolds of $\mathbb{K}\mathbf{H}^n$, it is clear that the restriction of the Riemannian metric to a horosphere is again a metric. When $t \to \infty$, the distance function that is induced by $e^{-2t} g^N_{(n,t)}$ converges to a Carnot-Carathéodory metric d_∞ on N.

Note that in the definition of d_∞ we made two choices, the choice of a base point $o \in \mathbb{K}\mathbf{H}^n$ and the choice of a geodesic through o, or equivalently of a point ∞ in the sphere at infinity. The left-invariance of the metric $g^N_{(n,t)}$ implies that if we change the base point to another base point that lies on the same horosphere centred at ∞, we do not change the Carnot-Carathéodory metric d_∞. However, changing the origin of the geodesic c leads to a proportional metric. This expresses the fact that the automorphism $e^{t\alpha}$ is a similarity of ratio e^t for d_∞.

It is less obvious to see how the choice of a different point $\infty \in \partial\mathbb{K}\mathbf{H}^n$ changes the metric.

Lemma 5.5. *When the point at infinity ∞ is changed, the Carnot-Carathéodory metrics d_∞ are conformal. In particular, the distribution Δ does not depend on the choice of ∞.*

These observations allow us to speak of quasiconformal transformations between boundaries of symmetric spaces, independently of the two choices made above. We refer the reader to [EHS93] for a proof of the previous lemma.

5.3 Quasiisometries and their extensions to the boundaries

In this section, we introduce quasiisometries and show that we can extend quasiisometries of rank-one symmetric spaces to their boundaries. From the fact that for each quasigeodesic there is a geodesic such that the geodesic and the quasigeodesic are contained in tubular neighbourhoods of each other, we deduce that two quasiisometries with the same extension to the boundary differ from each other only by a bounded amount. Finally, we show that the extension of a quasiisometry of a rank-one symmetric space is a quasiconformal map of the boundary.

Definition 5.6. Let X and X' be metric spaces with distance functions d and d' respectively. A *quasiisometric embedding* of X into X' is a map $f\colon X \to X'$ with two constants $L \geq 1$ and $C \geq 0$ such that for all $x, y \in X$ it holds that

$$-C + \frac{1}{L} d(x,y) \leq d'(f(x), f(y)) \leq Ld(x,y) + C.$$

A *quasiisometry* of X onto X' is given by a pair of quasiisometric embeddings $f\colon X \to X'$ and $g\colon X' \to X$ such that, for all $x \in X$ and $x' \in X'$, $d(g \circ f(x), x) \leq C, d'(f \circ g(X'), x') \leq C'$.
A *quasigeodesic* in X is a quasiisometric embedding of $\mathbb{R}$ into X.

Quasiisometries of hyperbolic spaces can be extended to the boundaries as follows. We let $f\colon \mathbb{K}\mathbf{H}^n \to \mathbb{K}\mathbf{H}^n$ be a (L, C)-quasiisometry. For all points $[c] \in \partial\mathbb{K}\mathbf{H}^n$ we set

$$f([c]) := [f(c)]. \tag{5.7}$$

We claim that this definition does not depend on the choice of geodesic $c \in [c]$. To prove this claim, we need an explicit expression of the distance on the boundary. Recall that there is no canonical Carnot-Carathéodory metric on the boundary, but that, by Lemma 5.5, different choices lead to conformal metrics, and that thus all choices induce the same topology. We claim that the following defines such a distance function. This is proven for example in [Bou95].

Fact 5.7. Let $[c], [c'] \in \partial\mathbb{K}\mathbf{H}^n$. We define

$$([c], [c'])_{(\infty, o)} := \frac{1}{2} \lim_{t\to\infty} (2t - d(c(t), c'(t))),$$

where d is the distance on $\mathbb{K}\mathbf{H}^n$. The distance on $\partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$ can then be defined as

$$d_\infty([c], [c']) = e^{-([c],[c'])_{(\infty,o)}}.$$

We can now confirm that the definition of $f\colon \partial\mathbb{K}\mathbf{H}^n \to \partial\mathbb{K}\mathbf{H}^n$ as in (5.7) does not depend on the choice of geodesic in $[c]$. Let therefore $c, c' \in [c]$. We conclude that

$$d_\infty([f(c)], [f(c')]) = e^{-([f(c)],[f(c')])_{(\infty,o)}} = \lim_{t\to\infty} e^{-\frac{1}{2}(2t - d(f(c(t)), f(c'(t))))}.$$

Since f is a (L, C)-quasiisometry, we have

$$e^{-\frac{1}{2}(2t+C-\frac{1}{L}d(c(t),c'(t)))} \leq e^{-\frac{1}{2}(2t-d(f(c(t)),f(c'(t))))} \leq e^{-\frac{1}{2}(2t-Ld(c(t),c'(t))-C)}, \tag{5.8}$$

and since $\lim_{t\to\infty} d(c(t), c'(t)) = 0$, it follows that $d_\infty([f(c)], [f(c')]) = 0$.

Remark 5.8. Note that (5.8) implies that we could as well have used quasigeodesics to define the extension of f to the boundary.

Remark 5.9. It follows from the definition of d_∞ that if a sequence of quasiisometric embeddings $f_i\colon \mathbb{K}\mathbf{H}^n \to \mathbb{K}\mathbf{H}^n$, where all f_i have the same constants L and C, converges uniformly, then the extensions $\bar{f}_i\colon \partial\mathbb{K}\mathbf{H}^n \to \partial\mathbb{K}\mathbf{H}^n$ converge uniformly as well.

Our next goal is to prove that two quasiisometries that extend to the same map on the boundary differ from each other only by a globally bounded amount. To do so, we need to see that for every quasigeodesic there exists a geodesic that differs from the quasigeodesic by a globally bounded distance. This is essentially a consequence of the negative curvature of the hyperbolic spaces. We omit a proof of this rather well-known result and refer the reader to [DK18, Theorem 11.72].

Lemma 5.10. *Let $\mathbb{K}\mathbf{H}^n$ be a hyperbolic space. For every quasigeodesic c of $\mathbb{K}\mathbf{H}^n$, there exists a geodesic c' contained in a tubular neighbourhood of c such that c is contained a tubular neighbourhood of c'. The width τ of these tubular neighbourhoods only depends on L and C.*

This lemma can be used to show that a quasiisometry is determined by its extension to the boundary at infinity up to transformations that move points by a bounded amount. Before we prove this result, we introduce the following notation.

Notation. For $a, b \in \mathbb{K}\mathbf{H}^n$ we denote by $[a, b]$ the geodesic segment with endpoints a and b.

Lemma 5.11. *Let $f, g\colon \mathbb{K}\mathbf{H}^n \to \mathbb{K}\mathbf{H}^n$ be two (L, C)-quasiisometries of a hyperbolic space. If f and g have the same extension to the sphere at infinity $\partial\mathbb{K}\mathbf{H}^n$, then for every $x \in \mathbb{K}\mathbf{H}^n$, we have $d(f(x), g(x)) \leq \tau'(L, C)$.*

Proof. Let $\tilde{g}$ be an inverse of g as in Definition 5.6, that is, there are constants C_1, C_2 such that

$$d(g \circ \tilde{g}(x), x) \leq C_1, \quad d(\tilde{g} \circ g(x), x) \leq C_2.$$

Thus, up to replacing f with the $(L^2, (L+1)C)$-quasiisometry $\tilde{g} \circ f$, we may assume that g is the identity map. Fix $x \in \mathbb{K}\mathbf{H}^n$ and let c be the geodesic line through x that is orthogonal to the geodesic segment $[x, f(x)]$. Being the composition of a quasiisometry and a geodesic,

$f \circ c$ is a quasigeodesic. By assumption, the quasigeodesic $f \circ c$ has the same extension to the sphere at infinity as c, so that it is asymptotic to c. By Lemma 5.10, there is a geodesic c' that is contained in a $\tau(L, C)$-tubular neighbourhood of $f \circ c$, and by transitivity, c and c' are asymptotic, thus $f \circ c$ lies in the $\tau'(L, C)$-neighbourhood of c. By orthogonality and since $x \in c$, we have $d(x, f(x)) = d(f(x), c)$, and it follows that $d(x, f(x)) \leq \tau'(L, C)$. □

Finally, we show that quasiisometries of rank-one symmetric spaces extend to quasiconformal maps of the boundary. This will allow us to reduce the proof of Theorem 8.1 to investigating the properties of quasiconformal maps between the Carnot groups that can be associated with the boundaries.

Proposition 5.12. *If f is a (L, C)-quasiisometry of a rank-one symmetric space $\mathbb{K}\mathbf{H}^n$, its extension to the boundary at infinity is η-quasiconformal, where η only depends on L and C.*

Proof. Consider the family of (L, C)-quasiisometries

$$\mathscr{F} = \{h \circ f \circ g \colon g \in \operatorname{Isom}(\mathbb{K}\mathbf{H}^n), h \in \operatorname{Isom}(\mathbb{K}\mathbf{H}^n)\}.$$

Let us choose a compact subset $K \subseteq \mathbb{K}\mathbf{H}^n$ and a point $x \in \mathbb{K}\mathbf{H}^n$, and consider the subfamily of elements of $\mathscr{F}$ which send x into K. As all elements of $\mathscr{F}$ are quasiisometries with the same constants, we can apply the Arzelà-Ascoli theorem to see that this subfamily is precompact in the topology of uniform convergence on compacts.

Recall from Remark 5.8 that uniform convergence on compacts of a family of quasiisometries of $\mathbb{K}\mathbf{H}^n$ implies uniform convergence on compacts of the family of extensions of the quasiisometries to the boundary in the topology of the boundary.

We translate the defining property of the subfamily of $\mathscr{F}$ that sends x into K to a property of the extensions to the boundary. To do so, we observe that the space of triples of distinct points in $\partial\mathbb{K}\mathbf{H}^n$ can be identified with the space of orthonormal pairs of tangent vectors of $\mathbb{K}\mathbf{H}^n$, which is the same as $\mathbb{K}\mathbf{H}^n$ as far as quasiisometries are concerned.

It follows that the extension of $\mathscr{F}$ between the spheres at infinity, which we call $\overline{\mathscr{F}}$, has the property that if K_1, K_2 and K_3 are three disjoint closed subspaces of $\partial\mathbb{K}\mathbf{H}^n$ and x_1, x_2, x_3 are three distinct points of $\mathbb{K}\mathbf{H}^n$, then the subfamily of F which sends x_i into K_i for all i is precompact.

To see that the extension of f to $\partial\mathbb{K}\mathbf{H}^n$, that we still denote f, is quasisymmetric, we need to prove that the distorsion

$$H_f(x) = \limsup_{r\to 0} \frac{\max\{|f(z+h)-f(z)| : |h|=r\}}{\min\{|f(z+h)-f(z)| : |h|=r\}}$$

of f is bounded at $x \in \partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$. Let N be the Carnot group associated with $\partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$. As N acts on itself by isometries, we may assume that x is the neutral element, otherwise we replace f by $f \circ L_{x^{-1}}$. Let us assume that $f(x) \neq \infty$. Again, one may assume that $f(x)$ is the neutral element of N, otherwise we replace f by $L_{f(x)^{-1}} \circ f$. Note that this is a priori not defined on ∞ and $(f \circ L_{x^{-1}})^{-1}(\infty)$, but can be continuously extended to include those two points.

For every $\varepsilon > 0$, let g_ε be an isometry of $\mathbb{K}\mathbf{H}^n$ that restricts to the automorphism $e^{t\alpha}$ on the horosphere centred at ∞. This horosphere can also be identified with N, as described in Section 5.2. We choose $t = t(\varepsilon)$ such that $e^{t\alpha}$ sends the unit k-annulus $a_k(x,1) = (B(x,1), B(x,k))$ on the k-annulus $a_k(x,\varepsilon) = (B(x,\varepsilon), B(x,k\varepsilon))$. Note that it is irrelevant what g_ε does on the rest of $\mathbb{K}\mathbf{H}^n$, since this is not visible on $\partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$.

Fix a point y in $\partial B(x,1)$. Let h_ε be an isometry of $\mathbb{K}\mathbf{H}^n$ that, similarly as g_ε, restricts to the automorphism $e^{-s\alpha}$ on a horosphere centred at ∞ such that $h_\varepsilon \circ f \circ g_\varepsilon(y) \in B(f(x),1)$. Hence, for $\varepsilon \to 0$, g_ε "zooms in", whereas h_ε "zooms back out".

When ε tends to 0, we have that $g_\varepsilon(y)$ tends to x. Hence h_ε has a dilating effect in the neighbourhood of $f(x)$. This shows that $h_\varepsilon(f(\infty))$ tends to ∞. We can now apply the criterion for precompactness of the family $\{h_\varepsilon \circ f \circ g_\varepsilon : 0 < \varepsilon < 1\}$ by choosing $x_1 = x, x_2 = y, x_3 = \infty$ and $K_1 = \{f(x)\}, K_2 = a_k(f(x),1), K_3 = \partial\mathbb{K}\mathbf{H}^n \setminus B(f(x),2k)$, where k is such that $\infty \in K_3$. Indeed, we have $h_\varepsilon \circ f \circ g_\varepsilon(x_i) \in K_i$. Note that this choice of K_1 only works because we assumed that x and $f(x)$ are the identity element of N.

We can repeat this argument for other points in $\partial B(x,1)$. Note that this may change h_ε, but we still reach the conclusion that the thus obtained subfamilies of $\overline{\mathscr{F}}$ are all precompact. It follows that the family of subsets $h_\varepsilon \circ f \circ g_\varepsilon(\partial B(x,1))$ is compact. Hence points in $h_\varepsilon \circ f \circ g_\varepsilon(\partial B(x,1))$ are a finite distance away from both $f(x)$ and ∞, which means that they are contained between two balls $B(f(x),r)$ and $B(f(x),R)$, where r and R both depend on the quasiisometry constants L and C. This is uniform with respect to x, so that f is η-quasisymmetric, and η depends only on L and C. Repeating this argument for f^{-1} completes the proof. □

6 Graded automorphisms of the boundaries at infinity

6.1 The cases of the quaternionic hyperbolic spaces and the octonionic hyperbolic plane

Recall that a 1-quasiconformal homeomorphisms is a quasiconformal homeomorphism whose differential is a similarity, that is, the product of a homothety $e^{t\alpha}$ and an isometry.

From Proposition 3.1 and Proposition 4.9 we know that the differential of any quasiconformal homeomorphism is an automorphism of a Carnot group. Since the exponential map is a diffeomorphism, this implies that there is a corresponding Lie algebra automorphism. The Lie algebra is graded, hence the automorphism preserves the gradation. We can make this precise.

Definition 6.1. Let N be a Carnot group whose Lie algebra $\mathfrak{n}$ has a gradation $\mathfrak{n} = V^1 \oplus V^2$. A *graded automorphism* of $\mathfrak{n}$ is given by two automorphisms $A \in \mathrm{Gl}_{\mathbb{R}}(V^1)$ and $B \in \mathrm{Gl}_{\mathbb{R}}(V^2)$ such that, if $x, y \in V^1$, we have

$$[Ax, Ay] = B[x, y].$$

In this section, we will see that in the quaternionic and the octonionic case, all graded automorphisms of $\mathfrak{n} = \mathrm{Lie}(N)$, where N is the Carnot group associated with the boundaries of the respective spaces, are similarities, which is not true for the boundaries of the real and complex hyperbolic spaces. This is essential to understanding why there is a clear difference in the properties of quasiisometries of the real and complex and the quaternionic and octonionic cases in Theorem 8.1. As always, our proof mainly follows Pansu's original proof, but in the last steps of the proof of Proposition 6.2, we have included additional explanations from [Bou18, Lemma 7.6].

Proposition 6.2. *Let $\mathbb{K}\mathbf{H}^n$ be a rank-one symmetric space and let N be the Carnot group associated with $\partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$. If $\mathbb{K}\mathbf{H}^n = \mathbb{H}\mathbf{H}^n$ or $\mathbb{K}\mathbf{H}^n = \mathbb{O}\mathbf{H}^2$, then any graded automorphism of $\mathfrak{n}$ is a similarity, that is, it is the product of an isometry of $\mathbb{K}^{n-1}$ and an automorphism $e^{t\alpha}$. Further, the group of graded automorphisms of $\mathfrak{n}$ coincides with the group of extensions of isometries of $\mathbb{K}\mathbf{H}^n$ that fix a geodesic.*

Recall that the Lie algebra $\mathfrak{n}$ of the groups N that we associate with the boundary of a rank-one symmetric space is nilpotent of class 2. It is graded by $\mathfrak{n} = \mathbb{K}^{n+1} \oplus \mathrm{Im}\,\mathbb{K}$ and its Lie bracket is defined by (5.3). We can endow the subspace $\mathbb{K}^{n-1}$ with the inner product

$$\langle x\,,\,y\rangle = \mathrm{Re}\,\left(\sum_{i=1}^{n-1} x_i\overline{y}_i\right). \tag{6.1}$$

A graded automorphism of $\mathfrak{n}$ is given by two automorphisms $A \in \mathrm{Gl}_{\mathbb{R}}(\mathbb{K}^{n-1})$ and $B \in \mathrm{Gl}_{\mathbb{R}}(\mathrm{Im}\,\mathbb{K})$ such that, if $x, y \in \mathbb{K}^{n-1}$, we have $[Ax, Ay] = B[x, y]$. To show that this is a similarity, we prove that A is the product of a homothety and an isometry of $\mathbb{K}^{n-1}$.

Consider the group D of graded automorphisms of $\mathfrak{n}$ with determinant equal to 1. This is sufficient since the full group of graded automorphisms is a semidirect product of D and $\{e^{t\alpha} : t \in \mathbb{R}\}$.

Lemma 6.3. *Let M be the group of extensions of isometries of $\mathbb{K}\mathbf{H}^n$ that fix a geodesic. Then M is a subgroup of D.*

Note that on $\partial\mathbb{K}\mathbf{H}^n$, these extensions are maps that fix a set of two points, namely the endpoints of the geodesic which the original maps of the hyperbolic spaces leave invariant.

Proof. Let f be an isometry on $\mathbb{K}\mathbf{H}^n$ that fixes a geodesic c, and choose $\infty = [c] \in \partial\mathbb{K}\mathbf{H}^n$. As shown in Section 5.2, upon fixing a base point $o \in \mathbb{K}\mathbf{H}^n$, we can identify $\mathbb{K}\mathbf{H}^n = N \rtimes \mathbb{R}$, with the Riemannian metric defined by (5.5) and (5.6). This metric is left-invariant on N. If we choose $o = c(0)$, then f has the form

$$f = (b, A_t) \in \mathrm{Aut}(N) \rtimes \{\text{translations on } c\}. \tag{6.2}$$

Let $\bar{f}$ be its extension to $\partial\mathbb{K}\mathbf{H}^n$. It is evident from (6.2) that $\bar{f} = b$. Since the exponential map of N is a global diffeomorphism, there is a unique Lie algebra homomorphism $\beta \in \mathrm{Aut}(\mathfrak{n})$ with $\exp(\beta(n)) = b(\exp(n))$ for all $n \in \mathfrak{n}$, and since f is an isometry, it follows from the form of $g_{(0,t)}$ that β is a graded automorphism. □

When $\mathbb{K} = \mathbb{H}$ we have $M = \mathrm{Sp}(n-1)\mathrm{Sp}(1)$. When $\mathbb{K} = \mathbb{O}$, then M is the subgroup of $\mathrm{O}(\mathbb{O})$ that is the image of Spin(7) in the spin representation. We will prove that $D = M$. To do so, first we observe that the Lie algebras are the same in the case of the hyperbolic planes, and later we extend this to the Lie groups. We treat the case of the quaternionic hyperbolic plane first. In the case that $N = \partial\mathbb{H}\mathbf{H}^2 \setminus \{\infty\}$, the Lie algebra of the group that contains the graded automorphisms of N with determinant equal to 1 is contained in $\mathfrak{sl}(4, \mathbb{R})$.

Lemma 6.4. *The Lie subalgebra $\mathfrak{so}(4)$ is maximal in $\mathfrak{sl}(4, \mathbb{R})$.*

Proof. Since $\mathfrak{sl}(4, \mathbb{R})$ consists of traceless 4×4-matrices and every matrix can be uniquely decomposed into a sum of a symmetric and an antisymmetric matrix, there are two subalgebras of symmetric traceless and antisymmetric matrices, denoted $S_0\mathbb{R}^4$ and $\Lambda^2\mathbb{R}^4$ respectively. Clearly $\Lambda^2\mathbb{R}^4$ is isomorphic to $\mathfrak{so}(4)$ as a Lie algebra. Any Lie subalgebra containing $\mathfrak{so}(4)$ is SO(4)-invariant. SO(4) acts on $S_0\mathbb{R}^4$ irreducibly, and it follows that such a Lie subalgebra of $\mathfrak{sl}(4,\mathbb{R})$ is either $\mathfrak{so}(4)$ or $\mathfrak{sl}(4,\mathbb{R})$. □

Since the Lie algebra of D cannot be all of $\mathfrak{sl}(4,\mathbb{R})$, and $\mathfrak{sp}(1)\oplus\mathfrak{sp}(1)\cong\mathfrak{so}(4)$, the Lie algebras of D and M are equal.

The second lemma treats the case of the octonionic hyperbolic plane. The Lie algebra of the group that contains the graded automorphisms of N with determinant equal to 1 in the case that $N=\partial\mathbb{O}\mathbf{H}^2\setminus\{\infty\}$ is contained in $\mathfrak{sl}(8,\mathbb{R})$.

Lemma 6.5. *The Lie subalgebras of* $\mathfrak{sl}(8,\mathbb{R})$ *containing* $\mathfrak{spin}(7)$ *are* $\mathfrak{sl}(8,\mathbb{R}),\mathfrak{so}(8)$ *and* $\mathfrak{spin}(7)$.

Proof. We decompose $\mathfrak{sl}(8,\mathbb{R})$ into irreducible components under Spin(7). By V we denote the spin representation of Spin(7). We can then view $\mathfrak{gl}(8)$ as $V^*\otimes V$.

The representation V of Spin(7) is associated to the fundamental weight $\bar{w}_3$. The representation $V^*\otimes V$ is isomorphic to $V\otimes V$, and it contains an irreducible component U with dominant weight $2\bar{w}_3$. An application of Weyl's character formula show that $\dim U=35$ [Bou95]. It is easy to see that the spin representation is a homomorphism of $\mathfrak{so}(7)$ into $\mathfrak{sl}(8,\mathbb{R})$ with irreducible image W whose dimension is 21.

There is another subspace of $\mathfrak{gl}(8)$ that we know to be invariant under Spin(7). This is the subspace of dimension 7 in $\mathfrak{sl}(8,\mathbb{R})$ spanned by elements of Im $\mathbb{O}$, acting by octonionic left-multiplication. Let us call this subspace Z. Clearly, this is not the trivial representation of Spin(7), and since Spin(7) has no nontrivial irreducible representation of dimension less than 7, it follows that Z is isomorphic to the natural representation of SO(7), and thus Z is irreducible.

We obtain the decomposition of $\mathfrak{sl}(8,\mathbb{R})$ in irreducible components $\mathfrak{sl}(8,\mathbb{R})=U\oplus W\oplus Z$. This leaves only four possibilities for a subspace of $\mathfrak{sl}(8,\mathbb{R})$ containing W. These are $W=\mathfrak{spin}(7)$, $W\oplus Z=\mathfrak{so}(8), W\oplus Z\oplus U=\mathfrak{sl}(8,\mathbb{R})$, or $U\oplus W$. We can prove that the last one is not a Lie subalgebra. Let us define

$$a=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad b=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad c=\frac{1}{2}[a,b]=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

We denote by A, respectively B and C, the 8×8-matrix that one obtains by concatenating four diagonal blocks equal to $\frac{1}{2}a$, respectively to b and c. Then $A, B \in U$ and $[A, B] = C \in Z$, since C is the matrix of the left multiplication by an element of Im $\mathbb{O}$ in the standard basis of the octonions. This finishes the proof that $U \oplus W$ is not a Lie subalgebra, so that the Lie subalgebras of $\mathfrak{sl}(8, \mathbb{R})$ containing $\mathfrak{spin}(7)$ are $\mathfrak{sl}(8, \mathbb{R}), \mathfrak{so}(8)$ and $\mathfrak{spin}(7)$. □

We remark that the Lie algebra of D in the case of the octonionic hyperbolic plane must thus be $\mathfrak{spin}(7)$, see for example [CS03, Chapter 8]. It follows that the Lie algebras of D and M are equal in the octonionic case, as well.

We now turn to the proof of Proposition 6.2. For the cases of the quaternionic and octonionic hyperbolic planes, we make use of the previously established equality of the Lie algebras of M and D.

Proof (*of Proposition 6.2 for the case* $\mathbb{H}\mathbf{H}^2$ *or* $\mathbb{O}\mathbf{H}^2$). From Lemma 6.4 and Lemma 6.5 we know that $\mathrm{Lie}(M) = \mathrm{Lie}(D)$. Given that M is a subgroup of D, we conclude that M is the connected component of D containing the identity. Conjugation by any element $d \in D$ is a continuous map on M, which we denote C_d, and since M is connected it follows that $C_d(M) \subseteq M$ for all $d \in D$, and thus D is contained in the normaliser of M that we denote $N_D(M)$.

The Centraliser-Normaliser theorem implies that ${N_D(M)}/{C_D(M)} \leqslant \mathrm{Aut}(M)$. Since M has no outer automorphisms, and the set of inner automorphisms of M is isomorphic to a subgroup of M, we have $D \leqslant C_D(M)M$. In both cases, M acts irreducibly on $\mathbb{K}$, hence by Schur's lemma we have $C_D(M) \subseteq \{+I, -I\} \subseteq M$, and $M = D$ follows. □

A different treatment is required to complete the proof for the other quaternionic cases.

Proof (*of Proposition 6.2 for the case* $\mathbb{H}\mathbf{H}^n$ *and* $n \geq 3$). Let A be a graded automorphism of $\mathfrak{n}$, that is, $A \in \mathrm{Gl}_{\mathbb{R}}(\mathbb{H}^{n-1})$ and there exists $B \in \mathrm{Gl}_{\mathbb{R}}(\mathrm{Im}\ \mathbb{H})$ such that $[Ax, Ay] = B[x, y]$. The simple calculation, where $x, y \in \mathbb{H}^{n-1}$ and $\lambda \in \mathrm{Im}\ \mathbb{H}$,

$$\langle y\, , \, \lambda x \rangle = \mathrm{Re} \sum_{j=1}^{n-1} y_j \overline{(\lambda x)_j} = \left(\mathrm{Im} \sum_{j=1}^{n-1} y_j \overline{x_j} \right) \bar{\lambda}$$

shows that $[x, y] = 0$ if and only if $y \perp x\mathrm{Im}\ \mathbb{H}$. As $(x\mathbb{H})^{\perp}$ is invariant when multiplied by imaginary numbers, any element y that commutes with $(x\mathbb{H})^{\perp}$ is orthogonal to $(x\mathbb{H})^{\perp}$, hence it is in $x\mathbb{H}$. This implies that for all $x \in \mathbb{H}^{n-1} \subseteq \mathfrak{n}$ such that $x \neq 0$, the quaternionic line $x\mathbb{H}$ is exactly the bicommutant of x, which is defined as $C_{\mathfrak{n}}\left(C_{\mathfrak{n}}(x)\right)$, the centraliser of the centraliser of x.

Now it is easy to see that A preserves the quaternionic lines, since A preserves the bicommutant of x for every $x \in N$. Let y such that $[x, y] = 0$. Let $z \in C_{\mathfrak{n}}(C_{\mathfrak{n}}(x)) = x\mathbb{H}$. Then

$$[y, Ax\lambda] = [y, Az] = \text{Im } y^*(Az),$$

so that A indeed preserves the quaternionic lines. The fundamental theorem of affine geometry (see for example [Ber87, Theorem 2.6.3]) implies that A is $\mathbb{H}$-skew linear, that is, there exists a ring automorphism σ of $\mathbb{H}$ such that for every $x \in \mathbb{H}^{n-1}$ and $\lambda \in \mathbb{H}, A(\lambda x) = \sigma(\lambda)A(x)$. Since the automorphisms of $\mathbb{H}$ are inner, σ is given by conjugation by μ for some $\mu \in \mathbb{H} \setminus \{0\}$. Real numbers commute with the quaternions, thus we may assume $|\mu| = 1$. We then have $\sigma \in \text{Sp}(1)$, and conclude that $A \in \text{Gl}_{\mathbb{H}}(n-1)\text{Sp}(1)$.

Up to composing A with an element of $\text{Sp}(n-1)$ that takes Ax to $\mathbb{H}x$, we may assume that A fixes a quaternionic line $\mathbb{H}x$. Note that the bracket $\Lambda^2\mathbb{H}x \to \text{Im } \mathbb{H}$ is surjective, consequently, $B \in \text{End}(\text{Im } \mathbb{H})$ is determined by the restriction of A to the line $\mathbb{H}x$.

Now consider $y = \mu x \in \mathbb{H}x$. Then $[x, y] = [x, \mu x] = \text{Im } \mu \, |x|^2$, where $|x|^2 = \langle x \, , \, x \rangle$, and

$$B\text{Im } \mu \, |x|^2 = B[x, y] = [Ax, A\mu x] = [Ax, \mu Ax] = \text{Im } \mu \, |Ax|^2 \, ,$$

so up to multiplying A on the right with a nonzero quaternion such that $|x| = |Ax|$, we may assume $B = id$. We show that $[Ax, Ay] = [x, y]$ implies that A is an isometry of $\mathbb{H}^{n-1}$.

The Lie bracket on V^1 can be expressed as

$$[x, y] = \omega_i(x, y)i + \omega_j(x, y)j + \omega_k(x, y)k, \tag{6.3}$$

where ω_α is a symplectic form on V^1 such that the above holds. These are given by

$$\omega_\alpha(x, y) = -\langle x \, , \, T_\alpha y \rangle,$$

where T_α is right multiplication by α for $\alpha \in \{i, j, k\}$.

Now let $A \in \text{Gl}_{\mathbb{R}}(\mathbb{K}^{m-1})$ such that $[Ax, Ay] = [x, y]$. For $\alpha \in \{i, j, k\}$ we then have that it must be $\omega_\alpha(Ax, Ay) = \omega_\alpha(x, y)$, and it follows that $A^*T_\alpha A = T_\alpha$. Taking the inverse on both sides and considering that $T_\alpha^{-1} = -T_\alpha$, we conclude $AT_\alpha A^* = T_\alpha$. It follows that

$$AT_iT_j = T_i(A^*)^{-1}T_j = T_iT_j.$$

Further, we see that $T_iT_j = T_k$, and thus

$$T_k = A^*T_kA = A^*AT_k,$$

which implies that $A^*A = id$. It follows that A is an isometry of $\mathbb{H}^{n-1}$, and then generally, we have $A \in \mathrm{Sp}(n-1)\mathrm{Gl}_{\mathbb{H}}(1)$. □

Corollary 6.6. *Let N be the Carnot group associated with the boundary at infinity of a quaternionic hyperbolic space $\mathbb{H}\mathbf{H}^n$ or the octonionic hyperbolic plane $\mathbb{O}\mathbf{H}^2$. Any quasiconformal homeomorphism between open subsets of N is 1-quasiconformal.*

Proof. Proposition 3.1 shows that any quasiconformal homeomorphism of such a Carnot group has a differential almost everywhere, and by Proposition 6.2, this differential is a similarity. □

6.2 The real and the complex case

Proposition 6.2 does not hold for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. This is why Theorem 8.1 cannot be extended to quasiisometries of the real and complex hyperbolic spaces. In this section, we provide concrete examples to demonstrate that not every quasiconformal homeomorphism of $\partial\mathbb{K}\mathbf{H}^n$ corresponds to a similarity for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

In the case of the real hyperbolic space, when the sectional curvature is constant, the group N is abelian and the derivation α is the identity, so that we have $e^{t\alpha} = e^t I_{n-1}$. Any linear map $A \in \mathrm{Gl}_{\mathbb{R}}(\mathbb{R}^{n-1}) \setminus \mathrm{CO}(n-1)$ is therefore a graded automorphism of N that is not a similarity. For example, for $n = 3$, if we choose $A = \mathrm{diag}(2, 1)$, we can define a map on $\mathbb{R}\mathbf{H}^3$ by setting

$$\begin{aligned} f \colon \mathbb{R}^2 \ltimes \mathbb{R} &\to \mathbb{R}^2 \ltimes \mathbb{R} \\ (n, a) &\mapsto (e^a An, a). \end{aligned}$$

It is easy to see that, up to choosing another base point in $\mathbb{R}\mathbf{H}^3$ and another point $\infty \in \partial\mathbb{R}\mathbf{H}^3$, the quasiisometry f extends to A on $\partial\mathbb{R}\mathbf{H}^3 \setminus \{\infty\}$, and that its differential, for example at the point $1 \in N \cong \partial\mathbb{R}\mathbf{H}^3 \setminus \{\infty\}$, is given by $Df(1) = A$. However, if we extend A to a map of the sphere S^2, for example by composing it with a stereographic projection and fixing the point ∞, we see that the resulting map is not conformal. This counterexample can easily be extended to the higher-dimensional real hyperbolic spaces. In the case of $\mathbb{R}\mathbf{H}^2$, an example illustrating that not every quasiconformal homeomorphism of the boundary corresponds to a similarity of N is given by any diffeomorphism of S^1 which stretches the sphere non-uniformly.

For the complex hyperbolic space $\mathbb{C}\mathbf{H}^n$, we have $\mathfrak{n} = \mathbb{C}^{n-1} \oplus i\mathbb{R}$ and the Lie bracket is the standard symplectic form on $\mathbb{C}^{n-1}$. The group of graded automorphisms is the conformal symplectic group $\mathrm{CSp}(2(n-1), \mathbb{R})$, which is larger than the group of similarities $\mathrm{U}(n-1) \ltimes \{e^{t\alpha} : t \in \mathbb{R}\}$. We remark that the subgroup of $\mathrm{Sp}(2(n-1), \mathbb{R})$ isomorphic to $\mathrm{U}(n-1)$ can be characterised as $\mathrm{Sp}(2(n-1), \mathbb{R}) \cap \mathrm{O}(2(n-1))$ [Arn89, p.225]. Hence, we can choose for example for $n = 2$,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{R}) \setminus \mathrm{U}(1).$$

Every $z \in V^2$ can be expressed as $z = [x, y]$ for some $x, y \in V^1$, so that A can be used to define a map $B : V^2 \to V^2$ by requiring that for $z = [x, y]$ we have $Bz = [Ax, Ay]$. We can thus define a graded automorphism on $\mathfrak{n}$ by setting

$$(A, B) : \mathbb{C} \oplus \mathbb{R} \to \mathbb{C} \oplus \mathbb{R}$$
$$(x, z) \mapsto (Ax, Bz).$$

Using exponential coordinates, this extends to a map of N that we call g. We further define a map on the hyperbolic space $\mathbb{C}\mathbf{H}^2$ by setting

$$f : N \ltimes \mathbb{R} \to N \ltimes \mathbb{R}$$
$$(n, a) \mapsto (g(e^{t\alpha} n), a).$$

As in the real case, up to choosing another base point in $\mathbb{C}\mathbf{H}^2$ and another point $\infty \in \partial\mathbb{C}\mathbf{H}^2$, we can easily see that the quasiisometry f extends to g on $\partial\mathbb{C}\mathbf{H}^2 \setminus \{\infty\}$, and that its differential, for example at $1 \in N \cong \partial\mathbb{C}\mathbf{H}^2 \setminus \{\infty\}$, is given by $Df(1) = g$. However, g is clearly not a similarity, because A is not an isometry of $\mathbb{C}$. This counterexample can be extended to the higher-dimensional complex hyperbolic spaces. It is also worth noting that further examples illustrating that not every quasiconformal homeomorphism of the boundary corresponds to a similarity of N are given by contact transformations of N, which are quasiconformal [KR85].

6.3 A geometric perspective

It is possible to view the proof of Proposition 6.2 in a more geometric way. We sketch the idea only for the complex and the quaternionic case, aiming to illustrate the difference between them. Our treatment can, in principle, be generalised to the boundaries of the real and the octonionic hyperbolic spaces, however, in the real case it is trivial and the octonionic case is less amenable than the others, so that we omit both.

First we observe that all arguments in the proof of Proposition 6.2 for the case $\mathbb{H}\mathbf{H}^n$ with $n \geq 3$ also apply to graded automorphisms of the boundaries of $\mathbb{C}\mathbf{H}^n$ up to the point where from $[Ax, Ay] = [x, y]$ for all $x, y \in V^1$ we derive that A is an isometry. We will therefore focus on such maps in our analysis and give the condition $[Ax, Ay] = [x, y]$ a geometric meaning.

Our analysis relies on lifts of curves in V^1 to horizontal curves in the Carnot group N associated with $\partial\mathbb{K}\mathbf{H}^{n+1} \setminus \{\infty\}$. Using exponential coordinates, we can construct a curve in N from a curve $\gamma \in V^1$ by inserting γ into the first $\dim V^1$ coordinates, and we will see that requiring horizontality already determines the remaining coordinates. In fact, these coordinates can be derived from the geometry of γ. This idea is inspired by Allcock's proof of an isoperimetric inequality for the Heisenberg groups in which horizontal lifts of curves in V^1 play a crucial role [All98]. We first make this precise and later explain how this relates to the proof of Proposition 6.2.

For better readability, we choose to consider $\partial\mathbb{K}\mathbf{H}^{n+1} \setminus \{\infty\}$ in this section. The highest index appearing in elements of $N = \mathbb{K}^n \ltimes \operatorname{Im} \mathbb{K}$ is then n instead of $n-1$.

We choose real parameterisations of $N = \mathbb{K}^n \times \operatorname{Im} \mathbb{K} \cong \mathfrak{n}$. In the complex case, we set

$$\begin{aligned} \psi \colon \mathbb{R}^{2n+1} &\to \mathbb{C}^n \times \operatorname{Im} \mathbb{C} \\ (x_1, \cdots, x_n, y_1, \ldots, y_n, z) &\mapsto ((x_1 + iy_1, \cdots, x_n + iy_n), iz)\,, \end{aligned}$$

and in the quaternionic case we take

$$\begin{aligned} \psi \colon \mathbb{R}^{4n+3} &\to \mathbb{H}^n \times \operatorname{Im} \mathbb{H} \\ (u_1, v_1, x_1, y_1, \cdots, u_n, v_n, x_n, y_n, z_i, z_j, z_k) &\mapsto ((u_l + iv_l + jx_l + ky_l)_{l=1}^n, iz_i + jz_j + kz_k)\,. \end{aligned}$$

Recall that with the exponential coordinates (2.1), the map ψ yields a global parameterisation of the corresponding Carnot groups N.

An inner product g_e on $\mathfrak{n}$ can be defined by declaring the basis vectors $\{\psi(b)\}$, where b runs through the orthonormal basis vectors of $\mathbb{R}^{n \cdot \dim_{\mathbb{R}} \mathbb{K} + \dim_{\mathbb{R}} \operatorname{Im} \mathbb{K}}$. We denote by X_1 the basis vector corresponding to the x_1-coordinate, and extend this nomenclature analogously to the other basis vectors. A metric on N can be defined by translating this inner product to other points p by left multiplication $g_p = (L_p)_* g_e$.

We identify the horizontal subspaces next. By (5.4), the horizontal subspace of T_eN is given by

$$H_e = \mathrm{span}_{\mathbb{R}}\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$$

in the complex case, whereas in the quaternionic case we have

$$H_e = \mathrm{span}_{\mathbb{R}}\{U_1, \cdots, U_n, V_1, \cdots, V_n, X_1, \cdots, X_n, Y_1, \cdots, Y_n\}.$$

It is translated to other points $p \in N$ by left-multiplication, hence $H_p = d_eL_pH_e \subseteq T_pN$.

Definition 6.7. Let γ be a curve in $V^1 \cong \mathbb{K}^n$. By x we denote the vector consisting of the first $n \cdot \dim_{\mathbb{R}} \mathbb{K}$ coordinates of (x, z). We define the projection $\pi\colon N \to \mathbb{K}^n, \exp(\psi(x, z)) \mapsto x$. A *horizontal lift* of γ is a horizontal curve $\tilde{\gamma}$ in N that satisfies the condition $\pi(\tilde{\gamma}) = \gamma$.

For the explicit construction of horizontal lifts, we introduce a characterisation of the horizontal subspaces as kernels of 1-forms. In the complex case, we set

$$\xi = \psi_*\eta, \qquad \text{and} \qquad \eta_{(x_1,y_1,\cdots,x_n,y_n,z)} = dz - \sum_{l=1}^{n} (x_l dy_l - y_l dx_l)\,,$$

so that $H_p = \ker \xi_p$. A curve $\beta\colon I \to N$ is horizontal if and only if $\beta'(t) \in \ker \xi_{\beta(t)}$ for all $t \in I$, where β' denotes the derivative of β with respect to the curve parameter.

The representation of the horizontal subspaces as kernels of the 1-form ξ yields a geometric interpretation for the z-coordinate, which we first present for the case $n = 1$. We use Stokes' theorem to see that for the projection onto the first two components of the horizontal path in coordinates $\sigma = \pi(\psi^{-1} \circ \beta)$, we have

$$\int_\sigma dz = \int_\sigma (xdy - ydx) = \int_{D_\sigma} d(xdy - ydx) = 2\int_{D_\sigma} dx \wedge dy = 2\mathrm{Area}(D_\sigma),$$

where D_σ is the area in $\mathbb{R}^2$ which is bounded by σ and line segments from the origin to the start- and endpoint of σ. This makes sense because the integral $\int_\lambda (xdy - ydx)$ vanishes whenever λ is a radial line segment so that we can always add it to σ to obtain a closed loop which bounds D_σ. For a horizontal path in coordinates, the z-coordinate therefore measures twice the area that its projection onto $\mathbb{R}^2$ encloses (potentially after closing it with radial line segments).

If we consider the general case, we need to adjust this interpretation slightly. An analogous argument shows that generally, the z-coordinate is given by the sum of the areas that the projections of γ in the (x_l, y_l)-planes enclose.

In the quaternion case, the horizontal subspaces can be described as the intersection of the kernels of three 1-forms, one for each of the (z_i, z_j, z_k)-directions. We denote these $\xi^{(i)}, \xi^{(j)}, \xi^{(k)}$, with the superscripts indicating the direction, and write them as a vector-valued differential form ξ, where $\xi_p = \left(\xi_p^{(i)}, \xi_p^{(j)}, \xi_p^{(k)}\right)^{\mathsf{T}}$. We set $\xi = \psi_* \eta$, with

$$\eta_{(v,w,x,y,z_i,z_j,z_k)} = \begin{pmatrix} \eta^{(i)}_{(v,w,x,y,z_i,z_j,z_k)} \\ \eta^{(j)}_{(v,w,x,y,z_i,z_j,z_k)} \\ \eta^{(k)}_{(v,w,x,y,z_i,z_j,z_k)} \end{pmatrix} = \begin{pmatrix} dz_i - udv + vdu - xdy + ydx \\ dz_j - udx + xdu - vdy + ydv \\ dz_k - udy + ydu - vdx + xdv \end{pmatrix},$$

so that the horizontal subspaces are $H_p = \ker \xi_p$.

Again, a curve $\beta\colon I \to N$ is horizontal if and only if $\beta'(t) \in \ker \xi_{\beta(t)}$ for all $t \in I$. By a similar argument as for the complex case, this can be interpreted geometrically as follows. In the case $n = 1$, the (z_i, z_j, z_k)-coordinates represent the following geometric quantities.

- The z_i-coordinate measures twice the sum of the areas in the (u, v)- and in the (x, y)-plane,
- The z_j-coordinate measures twice the sum of the areas in the (u, x)- and in the (v, y)-plane,
- The z_k-coordinate measures twice the sum of the areas in the (u, y)- and in the (v, x)-plane.

As in the complex case, this can be generalised to higher dimensions by considering sums of those areas of projections to subspaces.

Let $A \in \mathrm{Gl}_{\mathbb{R}}(V^1)$ satisfy $[Ax, Ay] = [x, y]$. Applying A^{-1} to an arbitrary curve γ and considering the horizontal lift $\widetilde{A^{-1}\gamma}$ yields a horizontal curve in N. We first consider the complex case and take $n = 1$. In view of (6.3), the z-coordinate of $\widetilde{A^{-1}\gamma}$ is identical with that of $\tilde{\gamma}$, but it is determined by the area that γ encloses in the plane spanned by $\{AX, AY\}$. However, the z-coordinate also represents the area that γ encloses in the plane spanned by $\{X, Y\}$, so that we can view A as an area-preserving transformation of V^1. Analogously, for arbitrary n we conclude that A is a transformation of $V^1 \cong \mathbb{R}^{2n} \cong C^n$ which leaves the sum of the areas that the projections of γ onto the (x_l, y_l)-planes enclose invariant. It follows that A is an element of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. This is clearly not the group of isometries of $\mathbb{C}^n$.

In the quaternionic case with $n = 1$, the (z_i, z_j, z_k)-coordinates of $\widetilde{A^{-1}\gamma}$ are identical with those of $\tilde{\gamma}$, but they are determined by sums of areas enclosed by projections of γ onto the two-dimensional subspaces spanned by pairs of basis vectors in the four-dimensional vector space $\mathrm{span}_{\mathbb{R}}\{U, V, X, Y\}$. By choosing curves for which one of the coordinates is constant, we further see that not only the sums of two of these areas are preserved, but the areas in each two-dimensional subspace separately. Moreover, we have $\det(A) = 1$ by assumption, thus A also preserves volumes. Requiring to simultaneously preserve volumes and the areas in two-dimensional subspaces spanned by pairs of basis vectors implies that A preserves the angles between the two-dimensional subspaces, and hence the Euclidean inner product of V^1 which is given by $\langle (u,v,x,y)\,,\,(u,v,x,y)\rangle = u^2+v^2+x^2+y^2$. In higher-dimensional cases, A preserves the sums of these on the (u_l, v_l, x_l, y_l)-subspaces. We recover as our preserved quantity the inner product (6.1), and it follows that A is an isometry of $\mathbb{H}^n$.

7 Realising 1-quasiconformal homeomorphisms as extensions of isometries

7.1 Outline of the proof

In this section, we prove that for every 1-quasiconformal homeomorphism there is an isometry of the corresponding hyperbolic space that extends to the same map on the boundary. This is our Proposition 7.4, and it is an important step in proving our main theorem because from Lemma 5.11 we know that two quasiisometries with the same extension to the boundary differ only by bounded amounts. By Proposition 6.2, in the case of the quaternionic hyperbolic spaces and the octonionic hyperbolic plane, all quasiconformal homeomorphisms of the boundary are in fact 1-quasiconformal, so that this will imply our main result.

The first step in proving Proposition 7.4 is to show that 1-quasiconformal homeomorphisms are locally Lipschitz. This is done by investigating the amount of deformation of spheres of radius R under f. When f is a global homeomorphism between Carnot groups, we let R tend to ∞. Doing so shows that f is globally Lipschitz.

In particular, this implies that if the differential of f at infinity is an isometry, then f is an isometry with respect to the Carnot-Carathéodory metric of the Carnot group associated with the boundary of a quaternionic hyperbolic space or the octonionic hyperbolic plane.

Our proof of the fact that every 1-quasiconformal homeomorphism of $\partial\mathbb{K}\mathbf{H}^n$ can be realised as the extension of an isometry of $\mathbb{K}\mathbf{H}^n$ makes use of this property in the following way. Given a 1-quasiconformal homeomorphism f, we can construct an isometry of $\mathbb{K}\mathbf{H}^n$ whose extension to the boundary has the same differential as f at the point ∞. In order to be able to translate the notion of differentiability, which we have established on Carnot groups, we need to translate between the boundary $\partial\mathbb{K}\mathbf{H}^n$ and the abstract group $N \cong \partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$. To do so, it is helpful to define embeddings of N into $\partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$ which fix a given other point in the boundary. This is done in Lemma 7.2.

We then prove that the map of the Carnot group N that we obtain from f is an isometry of the Carnot-Carathéodory metric, and by Lemma 7.3, we conclude that f is a group automorphism. The absolute continuity on lines of quasiconformal homeomorphisms allows us to conclude that whenever the differentials at ∞ of f and the extension of an isometry coincide, then the maps coincide as well, so that f is the extension of the isometry constructed.

7.2 All 1-quasiconformal homeomorphisms come from isometries

Recall that in Lemma 4.21 we have shown that capacities are invariant through 1-quasiconformal homeomorphisms. We use this to prove that 1-quasiconformal homeomorphisms between open subsets of Carnot groups are locally Lipschitz. This is done in the following lemma. For global homeomorphisms of the boundary of the rank-one symmetric spaces, we will see that this is true globally, which allows us to show that 1-quasiconformal homeomorphisms can be obtained as extensions of isometries.

Lemma 7.1. *Let $U \subseteq N$ and $V \subseteq N'$ be open subsets of the Carnot groups N and N'. Let $f\colon U \to V$ be a 1-quasiconformal homeomorphism. Then f is locally Lipschitz.*

Proof. Let $x \in U$. Fix $R < d(f(x), V)$. Set

$$D(x) = d(x, f^{-1}\partial B(f(x), R)).$$

For $\varepsilon < D(x)$, let us set $r(x) = \max\{d(f(x), f(z))\colon d(x,z) \leq \varepsilon\}$, so that $\mathrm{Lip}_f(x) = \lim_{\varepsilon\to 0} \frac{r(x)}{\varepsilon}$. The capacitor $C = f^{-1}B(f(x), R) \setminus \bar{B}(x, \varepsilon)$ is separated by the capacitor $S = B(x, D(x)) \setminus \bar{B}(x, \varepsilon)$, which is a spherical capacitor. Denoting its capacity by φ as in Remark 4.17, we have

$$\text{capacity } C \leq \text{capacity } S = \varphi\left(\frac{\varepsilon}{D(x)}\right).$$

Its image $f(C)$ separates the spherical capacitor $S' = B(f(x), R) \setminus B(f(x), r(x))$, so that

$$\text{capacity } f(C) \geq \text{capacity } S' = \varphi\left(\frac{r(x)}{R}\right).$$

From Lemma 4.21 we know that capacity $f(C) =$ capacity C, so that combining the two inequalities shows that $\varphi\left(\frac{r(x)}{R}\right) \le \varphi\left(\frac{\varepsilon}{D(x)}\right)$. The function φ is nondecreasing, which implies $\frac{r(x)}{R} \le \frac{\varepsilon}{D(x)}$, so that

$$\mathrm{Lip}_f(x) = \lim_{\varepsilon\to 0}\frac{r(x)}{\varepsilon} \le \frac{R}{D(x)}.$$

Since f is quasiconformal, $D(x)$ is nonzero, and since moreover $D(x) = d(x, f^{-1}\partial B(f(x), R))$ is continuous, the local dilation is bounded. Moreover, f is uniformly Lipschitz on every line on which it is almost everywhere uniformly continuous. We know from Proposition 4.15 that f is absolutely continuous on almost every line. This property is stronger than uniform continuity, so that f is locally Lipschitz. □

When f is a global homeomorphism between Carnot groups, we let the radius R tend to ∞. This lets us deduce that f is globally Lipschitz, with $\mathrm{Lip}_f(x) = \lim_{R\to\infty}\frac{R}{D(x)}$.

As a further preliminary result for the proof of Proposition 7.4 we define for every pair of points $x, \infty \in \partial\mathbb{K}\mathbf{H}^n$ an embedding $N \to \partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$ which maps 1 to x.

Lemma 7.2. *For any two distinct points $x, \infty \in \partial\mathbb{K}\mathbf{H}^n$, there exists a homeomorphism of Carnot groups $i_\infty\colon N \to \partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$ such that $i_\infty(1) = x$.*

Proof. The proof relies on the identification $\mathbb{K}\mathbf{H}^n \cong N \rtimes \mathbb{R}$ as in Section 5.2, and the extensions of isometries to the boundary, as in Section 5.3. Recall that the identification is not canonical, but depends on the choice of a base point in the hyperbolic space, and the choice of a point in the boundary. Our choice depends on the two points x and ∞.

Let c be a geodesic with endpoints x and ∞ so that $[c^-] = \infty$, where by c^- we denote the geodesic that we obtain from c by multiplying the curve parameter with -1. Choose $c(0)$ as a base point, however, note that the choice of base point on the geodesic c is irrelevant for this construction.

We construct an embedding $\imath_\infty$ of the abstract group N in $\mathrm{Isom}(\mathbb{K}\mathbf{H}^n)$, so that $\imath_\infty(1)$ fixes the geodesic c. We identify $\mathbb{K}\mathbf{H}^n$ with $N \rtimes \mathbb{R}$ with the base point $c(0)$ and the point at infinity ∞, so that N acts simply transitively on any horosphere centred at ∞ by left-multiplication. The Riemannian metric on $\mathbb{K}\mathbf{H}^n$ is such that this action is by isometries. By combining this with the trivial action on c, we obtain an isometry of $\mathbb{K}\mathbf{H}^n$.

We denote the resulting homeomorphism from N to $\mathrm{Isom}(\mathbb{K}\mathbf{H}^n)$ by $\imath_\infty$. It is easily seen that extensions of the isometries $\imath_\infty(n)$ for $n \in N$, denoted $\overline{\imath_\infty(n)}$, fix ∞, because the isometries map c^- to asymptotic geodesics (see for example [BH99, Lemma 10.26]). Hence, they allow us to identify $\partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$ and N through

$$i_\infty(g) = \overline{\imath_\infty(g)}(x),$$

where i_∞ is clearly bijective. Moreover, it follows that $i_\infty(1) = x$ because $\imath_\infty(1)$ fixes c. □

The next lemma is a partial result for proving that 1-quasiconformal homeomorphisms of the boundaries are, after translating them to maps of the abstract group N, group automorphisms.

Lemma 7.3. *Let f be a bijective isometry with respect to the Carnot-Carathéodory metric d_∞ of a Carnot group N that is identified with the boundary of a rank-one symmetric space, and assume that f fixes the neutral element. Then, there is a group automorphism which coincides with f on $N/[N,N]$ and commutes with the homotheties of N.*

Proof. A line in N is a horizontal curve whose projection onto $N/[N,N]$ is a curve that realises the distance between any two of its points. Isometries permute the lines. Let $z\Sigma$ be the union of the lines passing through z. We claim that $z \in [N,N]$ if and only if $z\Sigma \cap e\Sigma = \emptyset$. To see that this is true we observe that if $z \in [N,N]$ then there is no line passing through both z and the origin, which happens if and only if $z \in \exp(V^2) = [N,N]$.

Since f is an isometry, we have $f(z\Sigma) = f(z)\Sigma$, hence f preserves the centre. More generally, if we let π denote the projection from N to $N/[N,N]$, then for $c \in N/[N,N]$ any two $a, b \in \pi^{-1}(c)$ differ at most by $z = a^{-1}b$, where $z \in \mathcal{Z}(N)$, which we know is preserved by f. We consider the action of f on the set of fibres $\left\{\pi^{-1}(c) \colon c \in N/[N,N]\right\}$ and observe that f permutes the fibres. As f is invertible, f descends to a bijection of $N/[N,N]$ that we call $\bar{f}$.

Note that $N/[N,N] \cong \mathbb{K}^{n-1}$ has a vector space structure. Since f is an isometry, the map $\bar{f}$ preserves the Euclidean distance $d(u,v) = d_\infty(\pi^{-1}(u), \pi^{-1}(v))$. Thus $\bar{f}$ commutes with the homotheties of $N/[N,N]$, which are simply rescalings by a global factor e^t for some $t \in \mathbb{R}$. We use $\bar{f}$ to define an automorphism of N that we call $\bar{\bar{f}}$, by setting

$$\bar{\bar{f}}|_{N/[N,N]} = \bar{f}, \qquad \bar{\bar{f}}([a,b]) = [\bar{f}(\pi(a)), \bar{f}(\pi(b))] \quad \text{for } a, b \in N.$$

Per construction, $\bar{\bar{f}}^{-1} \circ f|_{N/[N,N]} = id_{N/[N,N]}$ and $\bar{\bar{f}}$ commutes with homotheties. □

With these three preliminary results, we can now prove that 1-quasiconformal homeomorphisms can be realised as extensions of isometries of the corresponding rank-one symmetric space.

Proposition 7.4. *Let $\mathbb{K}\mathbf{H}^n$ be a rank-one symmetric space, let $\partial\mathbb{K}\mathbf{H}^n$ be its sphere at infinity, endowed with the conformal structure defined in Section 5.2. For every 1-quasiconformal global transformation f of $\partial\mathbb{K}\mathbf{H}^n$ there is an isometry of $\mathbb{K}\mathbf{H}^n$ which extends to f on the boundary.*

Proof. We fix $x, \infty \in \partial\mathbb{K}\mathbf{H}^n$. The group $\mathrm{Isom}(\mathbb{K}\mathbf{H}^n)$ acts 2-transitively on $\partial\mathbb{K}\mathbf{H}^n$, that is, for all $(x,y),(x',y')$ there exists some $g \in N$ such that $g.x = x'$ and $g.y = y'$. We can thus assume that f fixes x and ∞.

By Lemma 7.2, there exist homeomorphisms $i_x\colon N \to \partial\mathbb{K}\mathbf{H}^n \setminus \{x\}$ and $i_\infty\colon N \to \partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$ that fix ∞ and x respectively. They allow us to identify $\partial\mathbb{K}\mathbf{H}^n \setminus \{x\}$ and $\partial\mathbb{K}\mathbf{H}^n \setminus \{\infty\}$ with N by

$$i_x(g) = \imath_x(g)\cdot\infty, \quad i_\infty(g) = \imath_\infty(g)\cdot x,$$

where both maps i_x, i_∞ are bijective.

Let us denote by A_t the translation by t along the geodesic from x to ∞. Being isometries, the maps A_t extend to the boundary, and we denote their extensions to $\partial\mathbb{K}\mathbf{H}^n$ still by A_t. We have

$$i_x \circ e^{t\alpha} = A_t \circ i_x, \quad i_\infty \circ e^{-t\alpha} = A_t \circ i_\infty. \tag{7.1}$$

These embeddings allow us to translate the theory of differentiation on Carnot groups that we developed in Section 2.3 to $i_\infty(N) \subseteq \partial\mathbb{K}\mathbf{H}^n$, where

$$Df(\infty) = \lim_{t\to\infty}(i_\infty \circ e^{t\alpha} \circ i_\infty^{-1}) \circ f \circ (i_\infty \circ e^{-t\alpha} \circ i_\infty^{-1}) = \lim_{t\to\infty} A_{-t} \circ f \circ A_t. \tag{7.2}$$

Any potential differential β is a similarity, that is, it is an automorphism of N that commutes with $e^{t\alpha}$. By bijectivity of the exponential map, we obtain a graded automorphism of $\mathfrak{n}$ that we call $\log(\beta) \in C_{Aut(\mathfrak{n})}(\alpha)$, which satisfies $\beta(\exp(n)) = \exp(\log(\beta)(n))$ for all $n \in \mathfrak{n}$. It is easy to see that β is indeed the differential of the automorphism of the boundary $i_\infty \circ \beta \circ i_\infty^{-1}$. By Proposition 6.2, we know to that this map is the extension of an isometry of $\mathbb{K}\mathbf{H}^n$.

We set $g = f \circ (i_\infty \circ \beta \circ i_\infty^{-1})^{-1}$, where $i_\infty \circ \beta \circ i_\infty^{-1} = Df(\infty)$. As β commutes with $e^{t\alpha}$, we have $Dg(\infty) = id_{\partial\mathbb{K}\mathbf{H}^n\setminus\{\infty\}}$. We can therefore, up to composing f with the extension of an isometry of $\mathbb{K}\mathbf{H}^n$, assume that $Df(\infty)$ is the identity.

To show that f itself is the extension of an isometry, we prove that $i_\infty^{-1} \circ f \circ i_\infty$ is a group automorphism of N. This, together with the absolute continuity of f on lines, implies that if $Df(\infty)$ is the identity, then so is f. As $f = i_\infty \circ \beta \circ i_\infty^{-1}$, and given that we already know that the latter is the extension of an isometry of $\mathbb{K}\mathbf{H}^n$, this concludes the proof.

To show that f is a group automorphism, we prove that f corresponds to a map on N that is an isometry of N with respect to the Carnot-Carathéodory metric defined in Section 5.2, and observe that f fixes the identity element of N. Using Lemma 7.3, we further show that an isometry that fixes the identity is already a group automorphism.

We can now use i_∞ to define a map $\bar{f}$ on N that corresponds to f, by setting $\bar{f} = i_\infty^{-1} \circ f \circ i_\infty$. Note that applying i_∞^{-1} here is well-defined because we assumed that f fixes ∞. It follows that

$$e^{t\alpha} \circ \bar{f} \circ e^{-t\alpha} \to id_N \quad \text{as } t \to \infty, \tag{7.3}$$

because we assumed that the differential of f at ∞ was the identity.

Our next goal is to prove that $\bar{f}$ is an isometry of N with respect to the distance d_∞ that was introduced in Section 5.2. Set $R = e^t$. When y is the neutral element of N, the number $D(y)$ that was introduced in Lemma 7.1 is such that

$$\begin{aligned}\frac{D(y)}{R} &= \frac{1}{R} d_\infty \left(y, \bar{f}^{-1} \partial B(\bar{f}(y), R)\right) = \frac{1}{R} d_\infty \left(y, \bar{f}^{-1} \circ e^{t\alpha} \partial B(\bar{f}(y), 1)\right) \\ &= d_\infty \left(e^{-t\alpha} y, e^{-t\alpha} \circ \bar{f}^{-1} \circ e^{t\alpha} \partial B(\bar{f}(y), 1)\right) = d_\infty \left(y, e^{-t\alpha} \circ \bar{f} \circ e^{t\alpha} \partial B(\bar{f}(y), 1)\right),\end{aligned}$$

where, in the second step, we choose $t = t(R)$ such that $e^{t\alpha}$ is a similarity of ratio R, and in the last step we used that $e^{-t\alpha} y = y$ because y is the neutral element. Then (7.3) shows that the right hand side tends to 1 when R, or equivalently t, tends to ∞. By Lemma 7.1 we have that $\mathrm{Lip}_f(y) \leq \frac{R}{D(y)}$, hence, for all $z \in N$,

$$d_\infty(\bar{f}(y), \bar{f}(z)) \leq d_\infty(y, z).$$

By precomposing $\bar{f}$ with translations, which are isometries of N, we obtain the same inequality for any y. Repeating this argument for $\bar{f}^{-1}$, we conclude that $\bar{f}$ is an isometry of (N, d_∞).

Note that f fixes x, which translates to $\bar{f}$ fixing the identity. As $\bar{f}$ is bijective, we can now apply Lemma 7.3 to obtain an automorphism of N that we call $\bar{\bar{f}}$. It commutes with the homotheties of N per construction, hence the differential of $\bar{\bar{f}}^{-1} \circ \bar{f}$ is easily seen to be the identity. With the property of absolute continuity on curves, we conclude that $\bar{\bar{f}}^{-1} \circ \bar{f}$ is itself the identity. Since $\bar{\bar{f}}$ is a group automorphism, so is $\bar{f}$. It follows that $\bar{f}$ and β coincide, and f is the extension of an isometry of $\mathbb{K}\mathbf{H}^n$. □

8 Proof of the main theorem

With the previous work it is now straightforward to prove our main result.

Theorem 8.1. *Every quasiisometry of quaternionic hyperbolic space $\mathbb{H}\mathbf{H}^n$, where $n \geq 2$, respectively of the octonionic hyperbolic plane $\mathbb{O}\mathbf{H}^2$, lies a bounded distance away from an isometry. That is, for every quasiisometry $q\colon \mathbb{K}\mathbf{H}^n \to \mathbb{K}\mathbf{H}^n$, where $\mathbb{K}\mathbf{H}^n = \mathbb{H}\mathbf{H}^n$ for $n \geq 2$ or $\mathbb{K}\mathbf{H}^n = \mathbb{O}\mathbf{H}^2$, there exists an isometry $f\colon \mathbb{K}\mathbf{H}^n \to \mathbb{K}\mathbf{H}^n$ and a map $g\colon \mathbb{K}\mathbf{H}^n \to \mathbb{K}\mathbf{H}^n$ such that $q = g \circ f$, and for some constant K, we have*

$$d(x, g(x)) \leq K \qquad \textit{for all } x \in \mathbb{K}\mathbf{H}^n.$$

Proof. Let f be a quasiisometry of quaternionic hyperbolic space $\mathbb{H}\mathbf{H}^n$ or the octonionic hyperbolic plane $\mathbb{O}\mathbf{H}^2$. By Proposition 5.12, f extends to a quasiconformal homeomorphism of the sphere at infinity. From Section 5.2 we know that the latter can, after removing one point, be identified with a Carnot group. The homeomorphism f is automatically 1-quasiconformal, so by Proposition 7.4 there exists an isometry $\tilde{f}$ which has the same extension to the sphere at infinity. Finally, by Lemma 5.11, f is a bounded distance from $\tilde{f}$, that is, $d(f(x), \tilde{f}(x))$ is bounded above, and the bound only depends on the quasiisometry constants L and C. □

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