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The Visual Boundaries of Hyperbolic Spaces

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Abstract. This thesis conducts an in-depth investigation into rank-one symmetric spaces and their visual boundaries. The analysis begins by exploring the projective models of hyperbolic spaces and the Riemannian structure of these. In this context, we examine geodesic rays and groups that act on the hyperbolic spaces transitively and by isometries. The visual boundary is introduced through an equivalence relation on the set of geodesic rays, and we show that it (minus one point) can also be identified with a subgroup of the above-mentioned groups and with any horosphere in the hyperbolic spaces. These three viewpoints each highlight different aspects of the visual boundary and allow us to define a structure that presents the visual boundaries of the hyperbolic spaces as subRiemannian manifolds. The final part focuses on the application of these concepts to a specific problem on the visual boundaries of the quaternionic hyperbolic spaces. Generalising a result by Allcock [All98], we construct homotopies from vertical line segments to horizontal paths, where we aim to control the area of the homotopies by a function of the length of the vertical line segments.

Abstrakt. Diese Arbeit untersucht symmetrische Räume vom Rang eins und ihre sichtbaren Ränder. Die Analyse beginnt mit der Erkundung der projektiven Modelle hyperbolischer Räume und ihrer Riemannschen Struktur. In diesem Zusammenhang untersuchen wir minimierende Geodäten und Gruppen, die auf die hyperbolischen Räume transitiv und isometrisch wirken. Der sichtbare Rand wird durch eine Äquivalenzrelation auf der Menge der minimierenden Geodäten eingeführt, und wir zeigen, dass er (ohne einen Punkt) auch mit einer Untergruppe der oben genannten Gruppen sowie mit jeder Horosphäre im hyperbolischen Raum identifiziert werden kann. Diese drei Perspektiven beleuchten jeweils verschiedene Aspekte des sichtbaren Rands und ermöglichen es uns, eine Struktur zu definieren, die die sichtbaren Ränder der hyperbolischen Räume als sub-Riemannsche Mannigfaltigkeiten darstellt. Der letzte Teil konzentriert sich auf die Anwendung dieser Konzepte auf ein spezifisches Problem auf den sichtbaren Rändern der quaternionischen hyperbolischen Räume. Ein Ergebnis von Allcock [All98] verallgemeinernd konstruieren wir Homotopien von vertikalen Geradensegmenten zu horizontalen Pfaden, wobei wir den Flächeninhalt der Homotopien durch eine Funktion der Länge der vertikalen Geradensegmente beschränken.

Contents

1 Introduction

This thesis explores rank-one symmetric spaces and their visual boundaries. Its goal is to achieve a description that enables to investigate their subRiemannian structure.

The non-compact irreducible rank-one symmetric spaces can be divided into three families, the real hyperbolic spaces, the complex hyperbolic spaces and the quaternionic hyperbolic spaces, and there is one exceptional case which is the octonionic hyperbolic plane [Bes78]. We only focus on the real, complex and quaternionic hyperbolic spaces and denote them as the $\mathbb{K}$-hyperbolic spaces $\mathbb{K}\mathbf{H}^n$, where $\mathbb{K}$ is either $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and $n \in \mathbb{N}$ indicates the dimension. To each $\mathbb{K}$-hyperbolic space we associate its visual boundary $\partial\mathbb{K}\mathbf{H}^n$. The visual boundary can be defined for a larger class of metric spaces and there are various reasons to study it [BH99], but in this thesis, we are interested in one specific aspect only which is that the visual boundary of a $\mathbb{K}$-hyperbolic space can be equipped with some structure that makes it a subRiemannian manifold. We set up this structure and use it to solve the following geometric problem on the visual boundary of the quaternionic hyperbolic space $\mathbb{H}\mathbf{H}^n$, that is inspired by a lemma by Allcock [All98, Lemma 4.3]. For subRiemannian manifolds, we can define horizontal and, in our case, also vertical paths. The problem under consideration is to map a vertical path of length L to a horizontal path of length $\sqrt{2\pi L}$ by a homotopy with an area that is bounded by $2L + \frac{8}{\sqrt{3}}\pi^{1/2}L^{3/2}$. We present a solution for special vertical paths.

This thesis is divided into three parts. The first part focuses on the $\mathbb{K}$-hyperbolic spaces. These spaces can be modelled in various ways. For our purpose, it is useful to start with the projective model. It realises the hyperbolic spaces as the subsets of the $\mathbb{K}$-projective spaces $\mathbb{K}\mathbf{P}^n$ for which a quadratic form Q of signature $(n, 1)$ is negative definite. We begin our investigation by defining the projective model, introducing the hyperbolic spaces as metric spaces. We then show that it is possible to obtain a Riemannian metric from the quadratic form Q, using that the $\mathbb{K}$-hyperbolic spaces can be realised as immersed submanifolds of $\mathbb{R}^m$ for some $m \in \mathbb{N}$. This construction is well-known for the real hyperbolic spaces, it yields the hyperboloid model. For the complex and quaternionic hyperbolic spaces, there is more arbitrariness in choosing an immersion, and we present one way to deal with this issue in the appendix. Our construction yields a model for the tangent spaces as well as a Riemannian metric on the hyperbolic spaces, and we prove that the metric space structure induced by the Riemannian metric coincides with the one from our initial definition.

In the course of deriving the induced metric space structure, we investigate geodesics in our hyperbolic spaces. We further introduce geodesic rays as unit speed geodesics, define the notion of two geodesic rays being asymptotic and answer the question under which conditions two geodesic rays are asymptotic. Moreover, we show that this property can be used to introduce an equivalence relation on the set of geodesic rays. This section lays important groundwork for the definition of the visual boundary.

We then determine for each $\mathbb{K}$-hyperbolic space a group $O_{\mathbb{K}}(Q)$ that acts transitively and by isometries on it. We start by identifying the group of matrices with entries in $\mathbb{K}$ preserving the quadratic form Q and we derive a useful characterisation of the matrices in this group. From the definition of the hyperbolic spaces through Q, it is easy to see that the induced action preserves the hyperbolic spaces.

Since this group action is transitive and by isometries, we can use the group to obtain a new perspective on our hyperbolic spaces. Moreover, we will later prove that, after subtracting one point, the visual boundaries of the hyperbolic spaces are identified with a subgroup N of $O_{\mathbb{K}}(Q)$. As our goal is the investigation of the structure of the visual boundary (minus one point) as a subRiemannian manifold, we determine the root space decomposition of the Lie algebra of $O_{\mathbb{K}}(Q)$, because it later provides convenient coordinates and a simple way of introducing a distribution which identifies horizontal curves in the subRiemannian manifold.

After this interjection, we return to investigating the group of isometries. We define subgroups N and A, where N is the group from above, and A is such that the product NA acts simply transitively on the hyperbolic spaces. This allows for the description of the hyperbolic spaces as semidirect products in terms of the matrix entries that, after fixing a base point, uniquely correspond to points in the hyperbolic spaces. We translate the Riemannian metric and the distance function into this description and show that the trace form on the Lie algebra of $O_{\mathbb{K}}(Q)$ yields essentially the Riemannian metric on the hyperbolic spaces. This concludes the discussion of the hyperbolic spaces.

In the second part, we define the visual boundary as the set of equivalence classes of asymptotic geodesic rays and investigate its form for the hyperbolic spaces. Thanks to our prior results, we know that we can characterise each equivalence class of geodesic rays in the projective model uniquely by a point at infinity to which the geodesic rays of this equivalence class converge. The identification of the elements of the visual boundary with points at infinity allows us to show that there is a one-to-one correspondence between the visual boundary (minus one point at infinity which we call pt) and N. Moreover, we prove in this section that N can also be identified with any horosphere centred at pt. This yields three objects that are identified, the visual boundary minus $\{\mathrm{pt}\}$, the group N and any horosphere centred at pt.

In the next section, we finally investigate the way in which the visual boundaries of the hyperbolic spaces (minus $\{\mathrm{pt}\}$) are subRiemannian manifolds. The identification with a horosphere yields a metric on the visual boundary of a hyperbolic space by restricting the Riemannian metric of the hyperbolic space to the horosphere. Each choice of horosphere yields a different metric, but these are equivalent in the sense that choosing a different such Riemannian metric changes statements about lengths at most by irrelevant constants. A distribution can be defined using a root space of the Lie algebra of N. It identifies the horizontal subspaces as the subspaces of the tangent spaces that are tangent to the distribution, while the vertical subspaces are defined as the subspaces of the tangent spaces that are perpendicular to the horizontal subspaces.

The last part of the thesis shows an application of the previously derived concepts. From this point on, the visual boundaries of different hyperbolic spaces are treated separately, because the vertical subspaces have different dimensions depending on the underlying hyperbolic space. There are no vertical subspaces for the real hyperbolic spaces, there are one-dimensional vertical subspaces for all complex hyperbolic spaces and three-dimensional vertical subspaces for all quaternionic hyperbolic spaces. We consider the complex case first and present a lemma by Allcock that provides a homotopy which takes vertical paths to horizontal paths and estimates the area of the homotopy [All98]. The result is used there to prove an isoperimetric inequality. Delving into the full context is beyond the scope of this thesis and we simply treat his lemma as a statement that is interesting for the purpose of working with a subRiemannian manifold. After presenting Allcock's lemma and his proof, we generalise it to the quaternionic case. The difference to the complex case is that there are now three vertical directions instead of one, which allows for many different vertical paths. We can use the homotopy from the complex case to solve this problem for paths along a line segment with only one nonzero coordinate, and we manipulate this solution to obtain a solution for general line segments. Our investigation results in the following theorem.

Theorem. *Any regular vertical path following a line segment in* $\partial\mathbb{H}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ *of length* L *is homotopic to a horizontal path of length* $\sqrt{2\pi L}$ *by a homotopy of area at most* $2L + 8\sqrt{\frac{\pi}{3}}L^{3/2}$.

2 The $\mathbb{K}$-hyperbolic spaces

2.1 Quaternions

Before we begin our investigation of the $\mathbb{K}$-hyperbolic spaces, we review the definition and some operations on the quaternions. The review follows Section 10.1.1 in [LD23].

The quaternions $\mathbb{H}$ are a four-dimensional algebra over $\mathbb{R}$. We declare $\{1, i, j, k\}$ as a basis that satisfies the following multiplication rules,

$$ij = k = -ji, \quad jk = i - kj, \quad ki = j = -ik, \quad i^2 = j^2 = k^2 = -1, \quad 1u = u1 = u$$

for all $u \in \mathbb{H}$. A quaternion can be expressed as

$$a + ib + jc + kd,$$

where $a, b, c, d \in \mathbb{R}$. In analogy to the complex numbers, we introduce the following:

- For $u = a + ib + jc + kd \in \mathbb{H}$, we define the *conjugate* $\overline{u} := a - ib - jc - kd$.
- For $u \in \mathbb{H}$ we define the *absolute value* $|u| := \sqrt{u\overline{u}}$.
- For $u \in \mathbb{H}$ we define its *real part* by $\mathrm{Re}\ u := \frac{u+\overline{u}}{2}$. If we write $u = a + ib + jc + kd$, then the real part of u is $\mathrm{Re}\ u = a$.
- For $u \in \mathbb{H}$ we define its *imaginary part* by $\mathrm{Im}\ u := \frac{u-\overline{u}}{2}$. If we write $u = a + ib + jc + kd$, then the imaginary part of u is $\mathrm{Im}\ u = ib + jc + kd$.

Note that unlike for the real and the complex numbers, multiplication in $\mathbb{H}$ is not commutative, and instead, for any $v, w \in \mathbb{H}$ it holds that

$$\overline{vw} = \overline{w}\ \overline{v}.$$

This can easily be verified using the representation of the quaternions v, w as a linear combination of basis elements and the respective multiplication laws. It follows that for matrices A, B, whose entries are quaternions, an identity for the Hermitian transposes A^*, B^* (which are defined in an analogous way as for matrices with complex entries) carries over from the case of complex matrices,

$$(AB)^* = B^*A^*.$$

We say that $x, y \in \mathbb{H}$ are *linearly dependent* if there exists some $\lambda \in \mathbb{H}$ such that $x\lambda = y$. We define linear dependence through right-multiplication with scalars instead of left-multiplication to ensure linearity of matrix-vector multiplication. This cannot be guaranteed if we choose left-multiplication because for any matrix A with quaternionic entries, we have $A(\lambda x) \neq \lambda Ax$ in general. However, as multiplication in $\mathbb{H}$ is associative, $A(x\lambda) = (Ax)\lambda$ holds, so that, with our definition of linear dependence, matrix multiplication is a linear operation.

2.2 Hyperbolic spaces

2.2.1 Hyperbolic spaces as metric spaces

In this section, we introduce the $\mathbb{K}$-hyperbolic spaces, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, through the projective model, following Chapter II.10 in [BH99] and Section 10 in [LD23]. We write $\mathbb{K}\mathbf{H}^n$ to denote the $\mathbb{K}$-hyperbolic space of dimension $\dim_{\mathbb{R}} \mathbb{K} \cdot n$ (as a manifold). In the projective model, the hyperbolic spaces are subsets of the projective spaces $\mathbb{K}\mathbf{P}^n$ for which a certain quadratic form is negative. We first recall projective spaces and then introduce the quadratic form and discuss its relevant properties before we define the hyperbolic space $\mathbb{K}\mathbf{H}^n$. Equipped with a distance function, the spaces $\mathbb{K}\mathbf{H}^n$ are metric spaces. We present this distance function as part of the definition, but it is in fact induced by a Riemannian metric on $\mathbb{K}\mathbf{H}^n$, as we show in Section 2.2.2.

We begin by recalling the definition of the projective spaces $\mathbb{K}\mathbf{P}^n$.

Definition 2.1. The *n-dimensional projective space over* $\mathbb{K}$ is the quotient

$$\mathbb{K}\mathbf{P}^n := \left(\mathbb{K}^{n+1} \setminus \{0\}\right)\big/\sim$$

where $(x_1, \cdots, x_{n+1}) \sim (y_1, \cdots, y_{n+1})$ if and only if there exists a $\lambda \in \mathbb{K} \setminus \{0\}$ such that $(x_1, \cdots, x_{n+1}) = (y_1\lambda, \cdots, y_{n+1}\lambda)$. The equivalence class of x is denoted $[x]$ and $(x_1, \cdots, x_{n+1})$ are *homogeneous coordinates.*

Note that the homogeneous coordinates are not really coordinates because many choices of homogeneous coordinates correspond to the same element in $\mathbb{K}\mathbf{P}^n$.

We define a quadratic form which can be used to obtain the hyperbolic spaces from the projective spaces. Rather than defining the quadratic form on $\mathbb{K}\mathbf{P}^n$, we define it on $\mathbb{K}^{n+1}$ and insert representatives of elements in $\mathbb{K}\mathbf{P}^n$, at the cost that we always have to confirm that our statements are independent of the chosen representative. The choice of quadratic form is not unique, and in fact, we could equivalently use any other quadratic form with signature $(n, 1)$ for this purpose,[1] but the

[1] See also Section B of the appendix, where we show that for any quadratic form of signature $(n, 1)$, there exists a basis in which it is diagonal with one eigenvalue -1 and n eigenvalues $+1$. Evidently, the hyperbolic spaces obtained from any such quadratic form are related to each other by an invertible transformation.

one defined below allows for a particularly convenient description later on.

Definition 2.2. On $\mathbb{K}^{n+1}$ we define the quadratic form $Q(x,y) := \langle x \,|\, y\rangle$ by

$$\langle x \,|\, y\rangle := \sum_{i=2}^{n} \overline{x}_i y_i - \overline{x}_1 y_{n+1} - \overline{x}_{n+1} y_1,$$

where $x = (x_1, \cdots, x_{n+1})$ and $y = (y_1, \cdots, y_{n+1})$.

We often write $\mathbb{K}^{n,1}$ to denote $\mathbb{K}^{n+1}$ endowed with the quadratic form Q in order to emphasise the additional structure on $\mathbb{K}^{n+1}$ that goes along with Q. For simplicity, we often write elements of $\mathbb{K}^{n+1}$ as row vectors without explicitly transposing them.

We establish some facts about $\langle \cdot \,|\, \cdot\rangle$.

Lemma 2.3. *The following are true.*

(a) *For all $x, y \in \mathbb{K}^{n,1}$ it holds that $\langle x \,|\, y\rangle = \overline{\langle y \,|\, x\rangle}$.*

(b) *The form $\langle \cdot \,|\, \cdot\rangle$ is additive in both arguments, and for all $x, y \in \mathbb{K}^{n,1}$ and $\lambda \in \mathbb{K}$ it holds that $\langle x \,|\, y\lambda\rangle = \langle x \,|\, y\rangle\lambda$ and $\langle x\lambda \,|\, y\rangle = \overline{\lambda}\langle x \,|\, y\rangle$.*

(c) *For all $x \in \mathbb{K}^{n,1}$ it holds that $\langle x \,|\, x\rangle \in \mathbb{R}$.*

(d) *For $x \in \mathbb{K}^{n,1}$ define $x^{\perp} := \{y \in \mathbb{K}^{n+1} \colon \langle x \,|\, y\rangle = 0\}$. Then for all $x \in \mathbb{K}^{n,1}$ with $\langle x \,|\, x\rangle < 0$, the restriction of $\langle \cdot \,|\, \cdot\rangle$ to $x^{\perp}$ is positive definite.*

Proof. The statements *(a)* and *(b)* immediately follow from the definition of the quadratic form. The property *(c)* is a direct consequence of *(a)* because it implies that $\overline{\langle x \,|\, x\rangle} = \langle x \,|\, x\rangle$. Finally, the proof of *(d)* is given in the appendix, see Proposition B.2. □

We are now ready to introduce the hyperbolic spaces through the projective model.

Definition 2.4. The $\mathbb{K}$-*hyperbolic n-space* is defined as the set

$$\mathbb{K}\mathbf{H}^n := \{[x] \in \mathbb{K}\mathbf{P}^n \colon \langle x \,|\, x\rangle < 0\}.$$

Equipped with the distance function defined by

$$\cosh^2 d([x],[y]) = \frac{\langle x \,|\, y\rangle\langle y \,|\, x\rangle}{\langle x \,|\, x\rangle\langle y \,|\, y\rangle}, \tag{2.1}$$

the $\mathbb{K}$-hyperbolic n-space $\mathbb{K}\mathbf{H}^n$ is a metric space.

It is easy to see that the condition $\langle x \,|\, x\rangle < 0$ in the definition of $\mathbb{K}\mathbf{H}^n$ does not depend on the chosen representative. If $x, y \in \mathbb{K}^{n,1} \setminus \{0\}$ such that $[x] = [y]$, then there exists a $\lambda \in \mathbb{K} \setminus \{0\}$ such that $x = y\lambda$. We calculate

$$\langle x \,|\, x\rangle = \langle y\lambda \,|\, y\lambda\rangle = \overline{\lambda}\langle y \,|\, y\rangle\lambda = |\lambda|^2 \langle y \,|\, y\rangle.$$

The last equality holds because $\langle y \,|\, y\rangle$ is real. With $\lambda \neq 0$ it follows that $\langle x \,|\, x\rangle < 0$ if and only if $\langle y \,|\, y\rangle < 0$.

Moreover, we prove that the distance function is well-defined and independent of the chosen representatives. This requires the following lemma.

Lemma 2.5. *(The Reverse Schwartz Inequality). If $\langle x \,|\, x\rangle < 0$ and $\langle y \,|\, y\rangle < 0$, then*

$$\langle x \,|\, y\rangle\langle y \,|\, x\rangle \geq \langle x \,|\, x\rangle\langle y \,|\, y\rangle.$$

Equality holds if and only if x and y are linearly dependent over $\mathbb{K}$.

Proof. If $y = x\lambda$ for some $\lambda \in \mathbb{K}$, then it is easy to see the equality

$$\langle x \,|\, y\rangle\langle y \,|\, x\rangle = \langle x \,|\, \lambda x\rangle\langle \lambda x \,|\, x\rangle = \langle x \,|\, x\rangle\overline{\lambda}\lambda\langle x \,|\, x\rangle = |\lambda|^2\,\langle x \,|\, x\rangle\langle x \,|\, x\rangle = \langle x \,|\, x\rangle\langle \lambda x \,|\, \lambda x\rangle = \langle x \,|\, x\rangle\langle y \,|\, y\rangle,$$

because $\langle x \,|\, x\rangle \in \mathbb{R}$.

If x and y are not linearly dependent, then y can be written as the sum of two vectors $y = v + u$ where $v \in x^{\perp}$ and $v \neq 0$, and $u \in \{x\alpha \colon \alpha \in \mathbb{K}\}$. From Lemma 2.3 *(d)* we know that the restriction of $\langle \cdot \,|\, \cdot\rangle$ to $x^{\perp}$ is positive definite, thus we have $u = x\alpha$ with $\alpha \neq 0$, because otherwise, it would not be possible to have $\langle y \,|\, y\rangle < 0$. It follows that $\langle x \,|\, y\rangle = \langle x \,|\, x\rangle\alpha \neq 0$. We choose $\lambda = -\frac{\langle x|x\rangle}{\langle x|y\rangle}$, then $\langle x + y\lambda \,|\, x\rangle = 0$, which implies that $x + y\lambda \in x^{\perp}$. The restriction of $\langle \cdot \,|\, \cdot\rangle$ to $x^{\perp}$ is positive definite, and because x and y are linearly independent, we have $x + y\lambda \neq 0$, so that $\langle x + y\lambda \,|\, x + y\lambda\rangle > 0$ follows. Writing out the last inequality and inserting $\lambda = -\frac{\langle x|x\rangle}{\langle x|y\rangle}$ yields

$$-\langle x \,|\, y\rangle\frac{\langle x \,|\, x\rangle}{\langle x \,|\, y\rangle} + \langle y \,|\, y\rangle\frac{\langle x \,|\, x\rangle^2}{\langle x \,|\, y\rangle\langle y \,|\, x\rangle} > 0.$$

Given that $\langle x \,|\, x\rangle < 0$, dividing the inequality by $\langle x \,|\, x\rangle$ reverses the inequality sign, so that we can rearrange it to

$$\langle x \,|\, y\rangle\langle y \,|\, x\rangle > \langle x \,|\, x\rangle\langle y \,|\, y\rangle.$$ □

It immediately follows from Lemma 2.5 together with statements *(a)* and *(b)* from Lemma 2.3 that the distance function (2.1) is well-defined and symmetric. Moreover, the facts that $\cosh(0) = 1$ and $\cosh(x) > 1$ for all $x \in \mathbb{R} \setminus \{0\}$ imply the positivity of d. It is not equally obvious that d satisfies a triangle inequality, and we refer to [BH99, Corollary II.10.9] for a proof.

2.2.2 Hyperbolic spaces as Riemannian manifolds

We defined $\mathbb{K}\mathbf{H}^n$ as a metric space by equipping it with the distance function d given by (2.1), but we can show that d is in fact induced by a Riemannian metric on $\mathbb{K}\mathbf{H}^n$. In the following, we aim to understand $\mathbb{K}\mathbf{H}^n$ as a Riemannian manifold as there are many advantages to adopting this perspective. For example, it enables us to view the hyperbolic spaces $\mathbb{K}\mathbf{H}^n$ as symmetric spaces and use their powerful tools. We begin our discussion by introducing a model for the tangent spaces that allows us to describe tangent vectors as elements of $\mathbb{K}^{n+1}$, which is useful for explicit calculations. After defining the Riemannian metric, we investigate geodesics in $\mathbb{K}\mathbf{H}^n$ and use them to prove that the distance function from our definition (2.1) coincides with the distance function induced by the Riemannian metric. A thorough understanding of the geodesics will also be important for the discussion of the visual boundaries.

Before we introduce the models of the tangent spaces, we point out that there is no canonical way to lift $\mathbb{K}\mathbf{H}^n$ into $\mathbb{K}^{n,1}$, thus there is no obvious concrete description of the tangent vectors as elements of $\mathbb{K}^{n+1}$. However, it is possible to do so in a non-canonical way, and this is presented in Section A of the appendix. In this section, we construct models for the tangent spaces identifying $T_{[x]}\mathbb{K}\mathbf{H}^n$ with $x^\perp$. Each representative of $[x]$ yields a different model, but we also show in the appendix that these models are equivalent in the sense that they allow the definition of a Riemannian metric which, when applying the substitution of representatives of the tangent vectors below, is invariant under changing the model. Following Chapter II.10 in [BH99], we present our result as a definition and refer the interested reader to the appendix for a derivation.

Definition 2.6. For $[x] \in \mathbb{K}\mathbf{H}^n$, we can identify the tangent space $T_{[x]}\mathbb{K}\mathbf{H}^n$ with the following set,

$$x^\perp = \left\{ y \in \mathbb{K}^{n+1} \colon \langle x \,|\, y \rangle = 0 \right\},$$

using the differential of the canonical projection $\pi \colon \mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{K}\mathbf{P}^n$.

If $u \in x^\perp$ is identified with $U \in T_{[x]}\mathbb{K}\mathbf{H}^n$, then u is the *tangent vector at x representing U*. For $\lambda \in \mathbb{K} \setminus \{0\}$, if u is the tangent vector at x representing U, then $u\lambda$ is the tangent vector at $x\lambda$ representing U.

Lemma 2.3*(d)* shows that the quadratic form Q is positive definite on $x^\perp$ for all $[x] \in \mathbb{K}\mathbf{H}^n$. Using the identification of $x^\perp$ with the tangent spaces of $\mathbb{K}\mathbf{H}^n$, this allows us to define a Riemannian metric on $\mathbb{K}\mathbf{H}^n$, thus turning $\mathbb{K}\mathbf{H}^n$ into a Riemannian manifold. We will later see that the induced distance function is the same as the distance function (2.1).

Definition 2.7. For $[x] \in \mathbb{K}\mathbf{H}^n$ and $u, v \in x^\perp$ representing tangent vectors $U, V \in T_{[x]}\mathbb{K}\mathbf{H}^n$, we set

$$g_{[x]}(U, V) = -\frac{\operatorname{Re} \langle u \,|\, v \rangle}{\langle x \,|\, x \rangle}. \tag{2.2}$$

Lemma 2.8 shows that $g_{[x]}$ is well-defined, independent of the chosen representative x of $[x]$ and that it yields a scalar product on $T_{[x]}\mathbb{K}\mathbf{H}^n$. We thus obtain a Riemannian metric g on $\mathbb{K}\mathbf{H}^n$ so that $(\mathbb{K}\mathbf{H}^n, g)$ becomes a Riemannian manifold.

Lemma 2.8. *For $[x] \in \mathbb{K}\mathbf{H}^n$ and $u, v \in x^\perp$, the map*

$$(u, v) \mapsto -\frac{\operatorname{Re} \langle u \,|\, v \rangle}{\langle x \,|\, x \rangle}$$

is a symmetric positive definite $\mathbb{R}$*-bilinear form. Moreover, for all* $\lambda \in \mathbb{K} \setminus \{0\}$*, it holds that*

$$\frac{\mathrm{Re}\ \langle u\lambda \,|\, v\lambda\rangle}{\langle x\lambda \,|\, x\lambda\rangle} = \frac{\mathrm{Re}\ \langle u \,|\, v\rangle}{\langle x \,|\, x\rangle}.$$

Proof. The $\mathbb{R}$-bilinearity and the symmetry of the map are obvious. From Lemma 2.3*(d)*, positive definiteness follows. The equation can immediately be derived from the definition of $\langle \cdot \,|\, \cdot\rangle$. □

Note that if we choose some $x \in \mathbb{K}\mathbf{H}^n$ with $\langle x \,|\, x\rangle = -1$, then the metric (2.2) is simply the real part of the restriction of $\langle \cdot \,|\, \cdot\rangle$ to $x^\perp$.

We finish the introduction of the hyperbolic spaces by showing that the induced distance function coincides with the distance function defined in (2.1). We prove this claim by constructing an arc length-parameterised geodesic from $[x]$ to $[y]$ for each pair of points $[x], [y] \in \mathbb{K}\mathbf{H}^n$. The distance of these points in terms of the induced distance function equals the difference of the curve parameter at the points. To do so, we first establish an expression for the geodesics in $\mathbb{K}\mathbf{H}^n$.

Lemma 2.9. *The geodesics* $\gamma\colon \mathbb{R} \to \mathbb{K}\mathbf{H}^n$ *are precisely the curves given by*

$$\gamma(t) = \left[x \cosh\left(\sqrt{\langle u \,|\, u\rangle} t\right) + \frac{u}{\sqrt{\langle u \,|\, u\rangle}} \sinh\left(\sqrt{\langle u \,|\, u\rangle} t\right)\right],$$

where $[x] \in \mathbb{K}\mathbf{H}^n$ *and* $u \in x^\perp$*, up to replacing* t *by* $\alpha t + t_0$ *for* $\alpha, t_0 \in \mathbb{R}$*.*

Proof. Let $[x] \in \mathbb{K}\mathbf{H}^n$ be represented by $x \in \mathbb{K}^{n,1}$ and $U \in T_{[x]}\mathbb{K}\mathbf{H}^n$ be represented by $u \in x^\perp$. We may take $\langle x \,|\, x\rangle = -1$, otherwise we replace x by $\frac{x}{\sqrt{|\langle x|x\rangle|}}$, but then is necessary to also replace u by $\frac{u}{\sqrt{|\langle x|x\rangle|}}$ so that it represents the same tangent vector U at $\frac{x}{\sqrt{|\langle x|x\rangle|}}$ as u represented at x. Note that this replacement changes the argument of cosh and sinh by a factor of $\frac{1}{\sqrt{|\langle x|x\rangle|}}$, but such a reparameterisation does not alter the fact that the curve defined below is a geodesic, as we will explain shortly. We claim that for

$$\sigma(t) = x \cosh\left(\sqrt{\langle u \,|\, u\rangle} t\right) + \frac{u}{\sqrt{\langle u \,|\, u\rangle}} \sinh\left(\sqrt{\langle u \,|\, u\rangle} t\right), \tag{2.3}$$

the curve $\gamma\colon t \mapsto= [\sigma(t)]$ is a geodesic. To prove the claim, we first confirm that γ is a well-defined curve in $\mathbb{K}\mathbf{H}^n$. This follows from

$$\begin{aligned}\langle \sigma(t) \,|\, \sigma(t)\rangle &= \langle x \,|\, x\rangle \cosh^2\left(\sqrt{\langle u \,|\, u\rangle} t\right) + \frac{\langle u \,|\, u\rangle}{\langle u \,|\, u\rangle} \sinh^2\left(\sqrt{\langle u \,|\, u\rangle} t\right) \\ &= \sinh^2\left(\sqrt{\langle u \,|\, u\rangle} t\right) - \cosh^2\left(\sqrt{\langle u \,|\, u\rangle} t\right) = -1.\end{aligned}$$

It remains to verify that the curve is a geodesic. Identifying $\mathbb{K}^{n+1}$ with $\mathbb{R}^{\dim_{\mathbb{R}} \mathbb{K}(n+1)}$, the Levi-Civita connection ∇ in this model is, by Proposition A.1 of the appendix, given by

$$\nabla = \mathrm{proj}_{T\mathbb{K}\mathbf{H}^n} D,$$

where D denotes the standard connection on $\mathbb{K}^{n+1}$ (defined by the identification with $\mathbb{R}^{\dim_{\mathbb{R}} \mathbb{K}(n+1)}$) and $\mathrm{proj}_{T\mathbb{K}\mathbf{H}^n}$ denotes the projection onto a model of the tangent bundle.[2] Using that $\gamma(t) = [\sigma(t)]$, we model the tangent spaces $T_{\gamma(t)}\mathbb{K}\mathbf{H}^n$ by $\sigma(t)^\perp$ for all t. We have

$$D_{\dot\sigma}\dot\sigma = \frac{d^2}{dt^2}\sigma(t) = \langle u \,|\, u\rangle \left(x\cosh\left(\sqrt{\langle u \,|\, u\rangle}t\right) + \frac{u}{\sqrt{\langle u \,|\, u\rangle}} \sinh\left(\sqrt{\langle u \,|\, u\rangle}t\right)\right) = \langle u \,|\, u\rangle\sigma(t),$$

which implies that

$$\nabla_{\dot\gamma}\dot\gamma = \mathrm{proj}_{\sigma(t)^\perp}\left(D_{\dot\sigma}\dot\sigma\right) = \mathrm{proj}_{\sigma(t)^\perp}(\langle u \,|\, u\rangle\sigma(t)) = 0,$$

and we conclude that γ is a geodesic.

We further prove that the geodesics cannot be of any other form. Let γ be a geodesic, set $[x] = \gamma(0)$ and choose a representative x with $\langle x \,|\, x\rangle = -1$. Moreover, we set $U = \dot\gamma(0)$ which is represented by $u \in x^\perp$. Per assumption, it is $\langle x \,|\, u\rangle = 0$. By the previous computation, the curve $\sigma(t) = x\cosh\left(\sqrt{\langle u \,|\, u\rangle}t\right) + \frac{u}{\langle u|u\rangle}\sinh\left(\sqrt{\langle u \,|\, u\rangle}t\right)$ defines a geodesic $\tilde\gamma$ in $\mathbb{K}\mathbf{H}^n$ by $\tilde\gamma\colon t \mapsto [\sigma(t)]$. It holds that $\tilde\gamma(0) = [x]$ and $\dot{\tilde\gamma}(0) = U$, and since the geodesic γ is completely determined by its initial point $\gamma(0)$ and tangent vector $\dot\gamma(0)$, up to rescaling the curve parameter and replacing x and u by $x\lambda$ and $u\lambda$ respectively for some $\lambda \in \mathbb{K} \setminus \{0\}$, the curves γ and $\tilde\gamma$ describe the same geodesic.

We specify what is a suitable rescaling of the curve parameter in this context. We claim that the replacement of t by $\alpha t + t_0$ for constants $\alpha, t_0 \in \mathbb{R}$ also yields a geodesic. To prove this claim, we write $\sigma'(t) = c(\alpha t + t_0)$ and $\gamma'(t) = [\sigma'(t)]$, and we see that

$$\nabla_{\dot\gamma'}\dot\gamma' = \mathrm{proj}_{T\mathbb{K}\mathbf{H}^n}\left(D_{\dot\sigma'}\dot\sigma'\right) = \mathrm{proj}_{\sigma'(t)^\perp}\Big(\langle u \,|\, u\rangle\sigma'(t)\Big) = \mathrm{proj}_{\sigma(\alpha t+t_0)^\perp}\Big(\alpha^2\langle u \,|\, u\rangle\sigma(\alpha t + t_0)\Big) = 0.$$

Therefore, γ' is also a geodesic. A straightforward computation shows that the rescaled curve

$$\sigma'(t) = x'\cosh\left(\sqrt{\langle u' \,|\, u'\rangle}(\alpha t + t_0)\right) + \frac{u'}{\sqrt{\langle u' \,|\, u'\rangle}}\sinh\left(\sqrt{\langle u' \,|\, u'\rangle}(\alpha t + t_0)\right)$$

can also be represented in the form (2.3) by setting

$$x = x'\cosh\left(\sqrt{\langle u' \,|\, u'\rangle}t_0\right) + \frac{u'}{\sqrt{\langle u' \,|\, u'\rangle}}\sinh\left(\sqrt{\langle u' \,|\, u'\rangle}t_0\right)$$

and

$$u = \alpha\left(x'\sinh\left(\sqrt{\langle u' \,|\, u'\rangle}t_0\right) + \frac{u'}{\sqrt{\langle u' \,|\, u'\rangle}}\cosh\left(\sqrt{\langle u' \,|\, u'\rangle}t_0\right)\right),$$

so that it is justified to assume that all geodesics are of the form (2.3). □

Equipped with an explicit expression for the geodesics in $\mathbb{K}\mathbf{H}^n$, we can determine the distance function d on the Riemannian manifold $(\mathbb{K}\mathbf{H}^n, g)$. The nomenclature for the induced distance

[2] More precisely, an isometric immersion τ is used to model $T\mathbb{K}\mathbf{H}^n$ as a subset of $T\mathbb{K}^{n+1}$, and $\mathrm{proj}_{T\mathbb{K}\mathbf{H}^n}$ projects onto the tangent bundle of the immersed submanifold $\tau(\mathbb{K}\mathbf{H}^n)$. The details are discussed in Section A of the appendix.

function is deliberately aligned with the one from the definition of $\mathbb{K}\mathbf{H}^n$ as a metric space because, as the next lemma shows, the two distance functions are identical.

Lemma 2.10. *The distance function d on $\mathbb{K}\mathbf{H}^n$ induced by the Riemannian metric* (2.2) *is given by*

$$\cosh^2 d([x],[y]) = \frac{\langle x \,|\, y\rangle\langle y \,|\, x\rangle}{\langle x \,|\, x\rangle\langle y \,|\, y\rangle}.$$

Proof. We prove the lemma by identifying for each pair of points $[x],[y] \in \mathbb{K}\mathbf{H}^n$ a geodesic from $[x]$ to $[y]$ that is parameterised by arc length. The arc length of the geodesic line segment from $[x]$ to $[y]$ then yields the distance of the two points.

We may assume that $\langle x \,|\, x\rangle = -1$, else we replace x by $\frac{x}{\sqrt{|\langle x|x\rangle|}}$ and u by $\frac{u}{\sqrt{|\langle x|x\rangle|}}$. We know from Lemma 2.9 that the geodesics in $\mathbb{K}\mathbf{H}^n$ with $\gamma(0) = [x]$ are of the form

$$\gamma(t) = \left[x\cosh\left(\sqrt{\langle u \,|\, u\rangle}\, t\right) + \frac{u}{\sqrt{\langle u \,|\, u\rangle}} \sinh\left(\sqrt{\langle u \,|\, u\rangle}\, t\right)\right] \qquad \text{for some } u \in x^{\perp}.$$

To obtain a geodesic that meets both $[x]$ and $[y]$, we need to choose u appropriately. We may also assume that $\langle y \,|\, y\rangle = -1$. We claim that there exists a unique $\lambda \in \mathbb{K}$ such that $|\lambda| = 1$ and $\langle x \,|\, y\lambda\rangle$ is real and negative. This can be seen by a simple calculation. Let $\alpha \in \mathbb{R}$ and $\alpha < 0$. From Lemma 2.5 we know that $|\langle x \,|\, y\rangle|^2 \geq \langle x \,|\, x\rangle\langle y \,|\, y\rangle = 1$. This implies $\langle x \,|\, y\rangle \neq 0$ so that we can set

$$\tilde{\lambda} = \frac{\alpha}{\langle x \,|\, y\rangle} \qquad \text{and} \qquad \lambda = \frac{\tilde{\lambda}}{|\tilde{\lambda}|}.$$

It is clear that $|\lambda| = 1$ and we have

$$\langle x \,|\, y\lambda\rangle = \langle x \,|\, y\rangle \frac{\alpha}{\langle x \,|\, y\rangle|\tilde{\lambda}|} = |\langle x \,|\, y\rangle| \frac{\alpha}{|\alpha|} = -\left|\langle x \,|\, y\rangle\right|,$$

which is real and negative by choice of α.

We can use λ to construct the geodesic joining $[x]$ and $[y]$. We set

$$a = \operatorname{arcosh}\left(-\langle x \,|\, y\lambda\rangle\right) \qquad \text{and} \qquad u = \frac{y\lambda - x\cosh a}{\sinh a}.$$

Note that a is well-defined because Lemma 2.5 implies that $|\langle x \,|\, y\lambda\rangle|^2 \geq \langle x \,|\, x\rangle\langle y \,|\, y\rangle\,|\lambda|^2 = 1$. Further, we calculate

$$\begin{aligned}
\langle u \,|\, u\rangle &= \frac{1}{\sinh^2 a}\Big(\langle y\lambda \,|\, y\lambda\rangle + \langle x \,|\, x\rangle\cosh^2 a - 2\mathrm{Re}\left(\langle x \,|\, y\lambda\rangle\right)\cosh a\Big)\\
&= \frac{1}{\sinh^2 a}\Big(-1 - \cosh^2 a - 2\langle x \,|\, y\lambda\rangle\cosh a\Big)\\
&= \frac{1}{\sinh^2 a}\Big(-1 - \langle x \,|\, y\lambda\rangle^2 + 2\langle x \,|\, y\lambda\rangle^2\Big)\\
&= \frac{1}{\sinh^2 a}\Big(-1 + \cosh^2 a\Big) = 1,
\end{aligned}$$

where we used that $\langle x \,|\, y\lambda \rangle$ is real, and

$$\langle x \,|\, u \rangle = \frac{1}{\sinh a} \left(\langle x \,|\, y\lambda \rangle - \langle x \,|\, x \rangle \cosh a \right) = \frac{1}{\sinh a} \left(\langle x \,|\, y\lambda \rangle - \langle x \,|\, y\lambda \rangle \right) = 0.$$

Let $\sigma(t) = x \cosh t + u \sinh t$. We define the curve $\gamma \colon t \mapsto [\sigma(t)]$, which, by Lemma 2.9, is a geodesic. Moreover, it is parameterised by arc length because

$$g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) = \langle \dot\sigma(t) \,|\, \dot\sigma(t) \rangle = \langle x \,|\, x \rangle \sinh^2 t + \langle u \,|\, u \rangle \cosh^2 t = 1 \qquad \text{for all } t \in \mathbb{R},$$

where we used that $\langle \sigma(t) \,|\, \sigma(t) \rangle = -1$ for all $t \in \mathbb{R}$. It remains to show that the curve meets both $[x]$ and $[y]$. This is easy to see because

$$\gamma(0) = [x] \quad \text{and} \quad \gamma(a) = \left[x \cosh a + \frac{y\lambda - x\cosh a}{\sinh a} \sinh a \right] = [y\lambda] = [y].$$

We conclude that the distance of $[x]$ and $[y]$ is given by the arc length of the geodesic line segment $\gamma([0,a])$, that is,

$$d([x],[y]) = \operatorname{Length}(\gamma([0,a])) = |a - 0| = \operatorname{arcosh}\left(|\langle x \,|\, y \rangle| \right).$$

Using $|\langle x \,|\, y \rangle| = \sqrt{\langle x \,|\, y \rangle \langle y \,|\, x \rangle}$, this can equivalently be written as

$$\cosh^2 d([x],[y]) = \cosh^2 \operatorname{Length}(\gamma([0,a])) = \langle x \,|\, y \rangle \langle y \,|\, x \rangle.$$

For $x, y \in \mathbb{K}^{n,1}$ with $\langle x \,|\, x \rangle$ and $\langle y \,|\, y \rangle$ not necessarily equal to -1, we replace x by $\tilde x = \frac{x}{\sqrt{|\langle x | x \rangle|}}$ (and likewise y) to be able to apply the above construction to $\tilde x$ and $\tilde y$. Using the linearity of $\langle \cdot \,|\, \cdot \rangle$, it follows that the distance of $[x]$ and $[y]$ in terms of x and y is given by

$$\cosh^2 d([x],[y]) = \frac{\langle x \,|\, y \rangle \langle y \,|\, x \rangle}{\langle x \,|\, x \rangle \langle y \,|\, y \rangle}. \qquad \square$$

2.3 Asymptotic geodesic rays

After the discussion of the geodesics in $\mathbb{K}\mathbf{H}^n$, this is a good point to introduce geodesic rays and asymptotic geodesic rays. They will be relevant later for defining the visual boundary.

Definition 2.11. A *geodesic ray* $c \colon [0,\infty) \to \mathbb{K}\mathbf{H}^n$ is a curve that satisfies

$$d(c(t), c(t')) = |t - t'| \qquad \text{for all } t, t' \in [0,\infty).$$

We say that c is the geodesic ray *issuing from* $[x]$ for some $[x] \in \mathbb{K}\mathbf{H}^n$ if $c(0) = [x]$.

Two geodesic rays $c, c' \colon [0,\infty) \to X$ are *asymptotic* if there exists a constant C_0 such that

$$d(c(t), c'(t)) \le C_0 \qquad \text{for all } t \ge 0.$$

We also say that c' is *asymptotic to* c if c and c' are asymptotic.

Up to reparametrisation, every geodesic is a geodesic ray, and every geodesic ray is obviously a unit speed geodesic. For us, the notions of a unit speed geodesic and a geodesic ray are therefore identical and we will sometimes add the attribute *unit speed* when speaking of a geodesic ray to emphasise the non-arbitrariness of its parameterisation. With this terminology, we can introduce an equivalence relation on the set of geodesic rays.

Lemma 2.12. *The relation on the set of geodesic rays that is defined by*

$$c \sim c' \quad :\Leftrightarrow \quad c, c' \text{ are asymptotic}$$

is an equivalence relation.

Proof. We only need to prove transitivity because reflexivity and symmetry are clear. Let therefore c, c', c'' be geodesic rays and assume that c and c' as well as c' and c'' are pairwise asymptotic, and that their distances are bounded by constants C_1 and C_2 respectively. The triangle equality implies

$$d(c(t), c''(t)) \leq d(c(t), c'(t)) + d(c'(t), c''(t)) \leq C_1 + C_2.$$

Hence, c and c'' are asymptotic and the relation defined in the statement of the lemma is an equivalence relation. □

We state the following lemma, which is a direct consequence of Proposition II.2.2 in [BH99], without proof.

Lemma 2.13. *Let $c, c' \colon [0, \infty) \to \mathbb{K}\mathbf{H}^n$ be two geodesic rays. Then the function*

$$t \mapsto d(c(t), c'(t))$$

is convex.

The next lemma gives a characterisation of asymptotic geodesic rays.

Lemma 2.14. *Two geodesic rays $c, c' \colon [0, \infty) \to \mathbb{K}\mathbf{H}^n$ given by*

$$c(t) = \left[x \cosh\left(\sqrt{\langle u \,|\, u\rangle} t\right) + \frac{u}{\sqrt{\langle u \,|\, u\rangle}} \sinh\left(\sqrt{\langle u \,|\, u\rangle} t\right)\right],$$
$$c'(t) = \left[y \cosh\left(\sqrt{\langle v \,|\, v\rangle} t\right) + \frac{v}{\sqrt{\langle v \,|\, v\rangle}} \sinh\left(\sqrt{\langle v \,|\, v\rangle} t\right)\right],$$

where $[x], [y] \in \mathbb{K}\mathbf{H}^n$ and $u, v \in x^\perp, y^\perp$ respectively, are asymptotic if and only if

$$\left[x + \frac{u}{\sqrt{\langle u \,|\, u\rangle}} \right] = \left[y + \frac{v}{\sqrt{\langle v \,|\, v\rangle}} \right]. \tag{2.4}$$

Moreover, given a geodesic ray c, then for each $[y] \in \mathbb{K}\mathbf{H}^n$ there is a unique geodesic ray issuing from $[y]$ that is asymptotic to c.

Proof. First we assume that (2.4) holds. We may take $x, y \in \mathbb{K}\mathbf{H}^n$ such that $\langle x \,|\, x\rangle = \langle y \,|\, y\rangle = -1$. Since geodesic rays are unit-speed parameterised, it must be $\langle u \,|\, u\rangle = \langle v \,|\, v\rangle = 1$. Therefore, we have $x + u = (y + v)\lambda$ for some $\lambda \in \mathbb{K}$ with $|\lambda| = 1$.

An explicit calculation shows that

$$\begin{aligned}\cosh^2 d(c(t), c'(t)) &= |\langle x\cosh(t) + u\sinh t \,|\, y\cosh(t) + v\sinh t\rangle|^2 \\ &= \left|\cosh^2 t\langle x \,|\, y\rangle + \cosh t\sinh t\,(\langle x \,|\, v\rangle + \langle u \,|\, y\rangle) + \sinh^2 t\langle u \,|\, v\rangle\right|^2.\end{aligned}$$

After inserting $v = (x + u)\lambda^{-1} - y$, the expression inside the absolute value becomes

$$\begin{aligned}&\cosh^2 t\langle x \,|\, y\rangle + \cosh t\sinh t\,(\langle x \,|\, v\rangle + \langle u \,|\, y\rangle) + \sinh^2 t\langle u \,|\, v\rangle \\ &= \cosh^2 t\langle x \,|\, y\rangle + \cosh t\sinh t\left(\langle x \,|\, x\rangle\lambda^{-1} + \langle x \,|\, u\rangle\lambda^{-1} - \langle x \,|\, y\rangle + \langle u \,|\, y\rangle\right) \\ &\qquad + \sinh^2 t\left(\langle u \,|\, x\rangle\lambda^{-1} + \langle u \,|\, u\rangle\lambda^{-1} - \langle u \,|\, y\rangle\right) \\ &= \langle x \,|\, y\rangle(\cosh^2 t - \cosh t\sinh t) + \langle u \,|\, y\rangle(\cosh t\sinh t - \sinh^2 t) + (-\cosh t\sinh t + \sinh^2 t)\lambda^{-1}.\end{aligned}$$

For the distance, we conclude that

$$\begin{aligned}&\cosh^2 d(c(t), c'(t)) \\ &= \left|\langle x \,|\, y\rangle(\cosh^2 t - \cosh t\sinh t) + \langle u \,|\, y\rangle(\cosh t\sinh t - \sinh^2 t) + (-\cosh t\sinh t + \sinh^2 t)\lambda^{-1}\right|^2 \\ &= |\langle x \,|\, y\rangle|^2\,(\cosh^2 t - \cosh t\sinh t)^2 + \left|\langle u \,|\, y\rangle + \lambda^{-1}\right|^2(\cosh t\sinh t - \sinh^2 t)^2 \\ &\qquad + (\cosh t\sinh t - \sinh^2 t)(\cosh^2 t - \cosh t\sinh t)\left(2\mathrm{Re}\ \langle x \,|\, y\rangle\langle u \,|\, y\rangle + 2\mathrm{Re}\ \langle x \,|\, y\rangle\lambda^{-1}\right).\end{aligned}$$

The terms $\langle u \,|\, y\rangle$ can be written in a simpler way, using $u = (v + y)\lambda - x$ and $\langle y \,|\, v\rangle = 0$, so that $\langle u \,|\, y\rangle = -\bar{\lambda} - \langle x \,|\, y\rangle$, where $\bar{\lambda} = \lambda^{-1}$ because $|\lambda| = 1$. Thus, we have

$$\begin{aligned}&\cosh^2 d(c(t), c'(t)) \\ &= |\langle x \,|\, y\rangle|^2\,(\cosh^2 t - \cosh t\sinh t)^2 + \left|-\lambda^{-1} - \langle x \,|\, y\rangle + \lambda^{-1}\right|^2(\cosh t\sinh t - \sinh^2 t)^2 \\ &\qquad + (\cosh t\sinh t - \sinh^2 t)(\cosh^2 t - \cosh t\sinh t)\left(2\mathrm{Re}\ \langle x \,|\, y\rangle(-\lambda^{-1} - \langle x \,|\, y\rangle) + 2\mathrm{Re}\ \langle x \,|\, y\rangle\lambda^{-1}\right) \\ &= |\langle x \,|\, y\rangle|^2\left((\cosh^2 t - \cosh t\sinh t)^2 + (\cosh t\sinh t - \sinh^2 t)^2\right. \\ &\qquad \left. + 2(\cosh t\sinh t - \sinh^2 t)(\cosh^2 t - \cosh t\sinh t)\right) \\ &= |\langle x \,|\, y\rangle|^2\left((\cosh^2 t - \cosh t\sinh t) + (\cosh t\sinh t - \sinh^2 t)\right)^2 \\ &= |\langle x \,|\, y\rangle|^2\left(\cosh^2 t - \sinh^2 t\right)^2 \\ &= |\langle x \,|\, y\rangle|^2.\end{aligned}$$

Hence $d(c(t), c'(t))$ is clearly bounded for all t, and we conclude that c and c' are asymptotic.

Now suppose that c and c' (defined as in the lemma) are asymptotic. We show that c' is the unique geodesic ray issuing from $[y]$ that is asymptotic to c. Suppose there is another geodesic ray c'' issuing from $[y]$ that is asymptotic to c. Then, by the transitive property of being asymptotic, c' and c'' are asymptotic, that is,

$$d(c'(t), c''(t)) \leq C_0$$

for some constant C_0. But then, since $d(c'(0), c''(0)) = 0$ and the distance function is never negative, Lemma 2.13 implies that

$$d(c'(t), c''(t)) = 0 \quad \text{for all } t \geq 0,$$

because a convex function that is bounded both above and below can only be constant. It follows that $c' = c''$ and therefore, the geodesic ray that issues from $[y]$ and is asymptotic to c is unique. $\square$

2.4 Isometry group

Since the spaces $\mathbb{K}\mathbf{H}^n$ are symmetric spaces, we can describe them with a group that acts on $\mathbb{K}\mathbf{H}^n$ transitively and by isometries, and we present this in Section 2.6. This section is devoted to discussing a suitable such group, which will be the group of invertible matrices preserving the quadratic form Q, referred to as $O_{\mathbb{K}}(Q)$. In the following, we define this group and show that the action of this group on $\mathbb{K}\mathbf{H}^n$ is well-defined and by isometries. We point out that this group is not *the* isometry group, but this is irrelevant and for the purposes of this thesis it is more convenient to work with a matrix group instead of the proper isometry group.

Following Section 10.4 in [LD23], we begin by defining the set of invertible matrices with entries in $\mathbb{K}$ and its subset $O_{\mathbb{K}}(Q)$. Let $\mathrm{Mat}(n+1, n+1; \mathbb{K})$ be the set of $(n+1) \times (n+1)$-matrices with entries in $\mathbb{K}$ and denote by $GL_{\mathbb{K}}(n+1)$ the subgroup of invertible matrices. The group $GL_{\mathbb{K}}(n+1)$ acts naturally on $\mathbb{K}^{n,1}$ by $\mathbb{K}$-linear automorphisms through matrix multiplication, where for $A = (a_{ij})_{i,j=1}^n$ and $(x_1, \cdots, x_{n+1}) \in \mathbb{K}^{n,1}$, we have

$$A.x = \left(\sum_{j=1}^{n+1} a_{1,j} x_j, \cdots, \sum_{j=1}^{n+1} a_{n+1,j} x_j \right).$$

Definition 2.15. We define the following subset of $GL_{\mathbb{K}}(n+1)$:

$$O_{\mathbb{K}}(Q) := \left\{ A \in GL_{\mathbb{K}}(n+1) \colon \langle A.x \mid A.y \rangle = \langle x \mid y \rangle \text{ for all } x, y \in \mathbb{K}^{n,1} \right\},$$

that is, $O_{\mathbb{K}}(Q)$ contains the invertible matrices preserving the quadratic form Q.

In the following, we present a sequence of propositions to establish the relevant properties of the group $O_{\mathbb{K}}(Q)$. To state the first proposition that provides a characterisation of $O_{\mathbb{K}}(Q)$, we introduce an alternative way of writing Q as a matrix-vector product. To this end, let

$$K := \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_{n-1} & 0 \\ -1 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}(n+1, n+1; \mathbb{K}),$$

where I_{n-1} is the $(n-1) \times (n-1)$-unit matrix. Then for any $x, y \in \mathbb{K}^{n,1}$, we can write

$$\langle x \,|\, y \rangle = x^* K y,$$

where x^* denotes the Hermitian conjugate of x.

Proposition 2.16. *A matrix $A \in GL_{\mathbb{K}}(n+1)$ is an element of $O_{\mathbb{K}}(Q)$ if and only if A satisfies $A^*KA = K$.*

Proof. First we assume that $A \in O_{\mathbb{K}}(Q)$. Then for all $x, y \in \mathbb{K}^{n,1}$ it holds that $x^*A^*KAy = x^*Ky$. By choosing all possible combinations of the canonical basis vectors for x, y, we conclude that $A^*KA = K$.

Now suppose that A satisfies $A^*KA = K$. Then $\langle Ax \,|\, Ay \rangle = x^*A^*KAy = x^*Ky = \langle x \,|\, y \rangle$, and it follows that $A \in O_{\mathbb{K}}(Q)$. □

Proposition 2.17. *The set $O_{\mathbb{K}}(Q)$ is a subgroup of $GL_{\mathbb{K}}(n+1)$.*

Proof. Clearly $I_{n+1} \in O_{\mathbb{K}}(Q)$, so $O_{\mathbb{K}}(Q)$ is nonempty. Proposition 2.16 provides a simple way to prove that $O_{\mathbb{K}}(Q)$ is closed under inversion and multiplication. Let therefore $A, B \in O_{\mathbb{K}}(Q)$. Then

$$(AB^{-1})^* KAB^{-1} = (B^{-1})^* A^* KAB^{-1} = (B^{-1})^* KB^{-1} = (B^{-1})^* B^* KBB^{-1} = K,$$

which shows that $AB^{-1} \in O_{\mathbb{K}}(Q)$. □

Proposition 2.18. *The induced action of $GL_{\mathbb{K}}(n+1)$ on $\mathbb{K}\mathbf{P}^n$ given by $A.[x] = [Ax]$ is well-defined.*

Proof. Let $\lambda \in \mathbb{K} \setminus \{0\}$ and $x \in \mathbb{K}^{n+1}$. We need to show that induced action of $GL_{\mathbb{K}}(n+1)$ on $\mathbb{K}\mathbf{P}^n$ is independent of the chosen representative, that is, $A.[x] = A.[x\lambda]$. This, however, is a direct consequence of the linearity of matrix multiplication, which implies that

$$(A(x\lambda))_i = \sum_{j=1}^{n+1} A_{ij}(x_j\lambda) = \left(\sum_{j=1}^{n+1} A_{ij}x_j \right) \lambda = (Ax)_i\lambda \quad \text{for all } i = 1, \cdots, n+1.$$ □

Proposition 2.19. *The induced action of $O_{\mathbb{K}}(Q)$ on $\mathbb{K}\mathbf{P}^n$ preserves the set $\mathbb{K}\mathbf{H}^n$.*

Proof. Let $x \in \mathbb{K}^{n,1}$ such that $[x] \in \mathbb{K}\mathbf{H}^n$. Then $\langle Ax \,|\, Ax \rangle = \langle x \,|\, x \rangle < 0$, because A preserves $\langle \cdot \,|\, \cdot \rangle$. This implies $A.[x] \in \mathbb{K}\mathbf{H}^n$. □

Proposition 2.20. *The action of $O_{\mathbb{K}}(Q)$ on $\mathbb{K}\mathbf{H}^n$ is by isometries.*

Proof. This follows immediately from the definition of the distance function (2.1) through $\langle \cdot \,|\, \cdot \rangle$. □

We emphasise that elements of the isometry group of $\mathbb{K}\mathbf{H}^n$ do not uniquely correspond to matrices from $O_{\mathbb{K}}(Q)$, because two matrices $A, B \in O_{\mathbb{K}}(Q)$ that are related by scalar multiples, $A = B\lambda$ for some $\lambda \in \mathbb{K} \setminus \{0\}$ (for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$), yield the same map on $\mathbb{K}\mathbf{H}^n$. Due to the non-commutativity of multiplication in the quaternions, matrices that are related by nonzero real scalar multiples describe the same map on $\mathbb{H}\mathbf{H}^n$, but this is no longer true for quaternionic scalars in general.

Taking into account that matrices in $O_{\mathbb{K}}(Q)$ satisfy the condition of Proposition 2.16, we conclude that two different matrices in $O_{\mathbb{R}}(Q)$ describe the same isometry if and only if one of them is a multiple of the other by a factor of -1, and the same holds true for matrices in $O_{\mathbb{H}}(Q)$, whereas in the complex case, any two matrices in $O_{\mathbb{C}}(Q)$ that are multiples by a unit complex number describe the same isometry. This is why $O_{\mathbb{K}}(Q)$ is not *the* isometry group.

2.5 Root space decomposition

In this section we determine the Lie algebra of $O_{\mathbb{K}}(Q)$ and its restricted root space decomposition. We will see that there are only four restricted roots in the complex and the quaternionic cases, and only two in the real case. In Section 3, we prove that a suitably chosen subgroup of $O_{\mathbb{K}}(Q)$ can be identified with the visual boundary of the spaces $\mathbb{K}\mathbf{H}^n$ (minus one point), and that the exponential map provides a diffeomorphism between the direct sum of two root spaces and this subgroup. This allows us to choose coordinates for the visual boundary (minus one point) using a basis of the root spaces by identifying each element with its preimage under the exponential map. Determining the restricted root spaces is therefore an important step in deriving convenient coordinates.

In order to establish the language and notation used here, we review some concepts for describing symmetric spaces. Recall that if G is a connected group acting on $\mathbb{K}\mathbf{H}^n$ transitively and by isometries, $p \in \mathbb{K}\mathbf{H}^n$ and $\sigma\colon G \to G$ is an involutive automorphism with $(G^\sigma)_\circ \subseteq \mathrm{Stab}_G(p) \subseteq G^\sigma$, where $G^\sigma = \mathrm{Fix}(\sigma) = \{g \in G\colon \sigma(g) = g\}$ and $\mathrm{Ad}(\mathrm{Stab}_G(p)) < \mathrm{GL}(\mathfrak{g})$ is compact, then we have $\mathbb{K}\mathbf{H}^n \cong {}^{G}\!/_{\mathrm{Stab}_G(p)}$ isometrically. In Lie algebra terms, for $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{k} = \mathrm{Lie}(\mathrm{Stab}_G(p))$ and $d\sigma = \Theta$, the Cartan involution corresponding to p, the tuple $(\mathfrak{g}, \Theta)$ is an orthogonal symmetric Lie algebra. In particular, it is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = E_1(\Theta)$ and $\mathfrak{p} = E_{-1}(\Theta)$ are the (± 1)-eigenspaces of Θ, and we have $\mathfrak{p} \cong T_p\mathbb{K}\mathbf{H}^n$ [Hel78].

From Proposition 2.20 we know that $O_{\mathbb{K}}(Q)$ acts on $\mathbb{K}\mathbf{H}^n$ by isometries and in Proposition 2.28 we will see that this action is transitive, so that we can set $G = O_{\mathbb{K}}(Q)_\circ$, which denotes the connected component of $O_{\mathbb{K}}(Q)$ containing the identity, here. Note that for $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$, the groups $O_{\mathbb{K}}(Q)$ are precisely $U(n,1)$ respectively $Sp(n,1)$. These are already connected, so that we have $G = U(n,1)$ and $G = Sp(n,1)$. For $\mathbb{K} = \mathbb{R}$, it holds that $O_{\mathbb{R}}(Q) = O(n,1)$ which has four connected components, so that we have $G = O(n,1)_\circ$, which is a proper subset of $O_{\mathbb{K}}(Q)$, in the real case [Kna96].

We start by determining the Lie algebra $\mathfrak{g}$ of G. From Proposition 2.16 we know that $A \in O_{\mathbb{K}}(Q)$ if and only if $A^*KA = K$, where

$$K = \begin{pmatrix} & & -1 \\ & I_{n-1} & \\ -1 & & \end{pmatrix}.$$

Differentiation of the condition $A^*KA = K$ yields a characterisation of $\mathfrak{g}$,

$$\mathfrak{g} = \{A \in \mathrm{Mat}(n+1, n+1; \mathbb{K})\colon A^*K + KA = 0\}.$$

Considering the structure of K, it is natural to divide $A \in \mathfrak{g}$ into a block matrix as follows

$$A = \begin{pmatrix} \psi & b & \gamma \\ d & E & f \\ \sigma & h & \tau \end{pmatrix},$$

where $E \in \mathrm{Mat}(n-1, n-1; \mathbb{K})$, the elements $b^\mathsf{T}, d, f, h^\mathsf{T}$ are in $\mathbb{K}^{n-1}$ and $\psi, \gamma, \sigma, \tau \in \mathbb{K}$. We derive the criteria on the entries of A under which the condition $A^*K + KA = 0$ is satisfied. We find that

$$A = \begin{pmatrix} \psi & b & \gamma \\ d & E & f \\ \sigma & h & \tau \end{pmatrix} \in \mathfrak{g} \quad \text{if and only if} \quad \begin{cases} \psi + \tau^* = 0, & b^* + f = 0, \\ \gamma + \gamma^* = 0, & d^* + h = 0, \\ \sigma + \sigma^* = 0, & E^* + E = 0. \end{cases}$$

We conclude that $\mathfrak{g}$ is of the form

$$\mathfrak{g} = \left\{ \begin{pmatrix} \psi & b & \gamma \\ d & E & b^* \\ \sigma & d^* & -\psi^* \end{pmatrix} : \psi \in \mathbb{K}, \gamma, \sigma \in \mathrm{Im}\ \mathbb{K}, E = -E^*, b^\mathsf{T}, d \in \mathbb{K}^{n-1} \right\},$$

where we set $\mathrm{Im}\ \mathbb{R} = \{0\}$. We choose the Cartan involution

$$\Theta \colon \mathfrak{g} \to \mathfrak{g}, \quad X \mapsto -X^*.$$

Then the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ consists of

$$\mathfrak{k} = \left\{ \begin{pmatrix} \psi & b & \gamma \\ -b^* & E & b^* \\ \gamma & -b & \psi \end{pmatrix} : \psi \in \mathrm{Im}\ \mathbb{K}, E^* = -E, b^\mathsf{T} \in \mathbb{K}^{n-1}, \gamma \in \mathrm{Im}\ \mathbb{K} \right\},$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} \psi & b & \gamma \\ b^* & 0 & b^* \\ -\gamma & b & -\psi \end{pmatrix} : \psi \in \mathbb{R}, b^\mathsf{T} \in \mathbb{K}^{n-1}, \gamma \in \mathrm{Im}\ \mathbb{K} \right\}.$$

We claim that the Cartan involution Θ corresponds to the point $o = [(1, 0, \cdots, 0, 1)]$. To see that this is true, we choose $\psi \in \mathrm{Im}\ \mathbb{K}, b^\mathsf{T} \in \mathbb{K}^{n-1}$, $E \in \mathrm{Mat}(n-1, n-1; \mathbb{K})$ such that $E^* = -E$ and $\gamma \in \mathrm{Im}\ \mathbb{K}$, and compute

$$\begin{pmatrix} \psi & b & \gamma \\ -b^* & E & b^* \\ \gamma & -b & \psi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \psi + \gamma \\ 0 \\ \vdots \\ 0 \\ \psi + \gamma \end{pmatrix},$$

from which we conclude that

$$\exp \begin{pmatrix} \psi & b & \gamma \\ -b^* & E & b^* \\ \gamma & -b & \psi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{\psi+\gamma} \\ 0 \\ \vdots \\ 0 \\ e^{\psi+\gamma} \end{pmatrix}.$$

It follows that $\exp(\mathfrak{k})$ stabilises $[(1, 0, \cdots, 0, 1)]$.

To determine the root space decomposition, a maximal abelian subalgebra $\mathfrak{a}$ is needed. We claim that we can take

$$\mathfrak{a} = \operatorname{span}_{\mathbb{R}} \{a\}, \qquad \text{where } a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and we will later see that $\mathfrak{a}$ is maximal. The ansatz $\mathrm{ad}_a(X) = \alpha(a)X$ yields the restricted roots

$$\Lambda = \{-2\alpha, -\alpha, \alpha, 2\alpha\},$$

where $\alpha \in \mathfrak{a}^*$, the dual space of $\mathfrak{a}$, such that

$$\alpha(a) = 1.$$

The corresponding restricted root spaces are

$$\mathfrak{g}_{-2\alpha} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma & 0 & 0 \end{pmatrix} : \sigma \in \operatorname{Im} \mathbb{K} \setminus \{0\} \right\},$$

$$\mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ d & 0 & 0 \\ 0 & d^* & 0 \end{pmatrix} : d \in \mathbb{K}^{n-1} \setminus \{0\} \right\},$$

$$\mathfrak{g}_{\alpha} = \left\{ \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & b^* \\ 0 & 0 & 0 \end{pmatrix} : b^{\mathsf{T}} \in \mathbb{K}^{n-1} \setminus \{0\} \right\},$$

$$\mathfrak{g}_{-2\alpha} = \left\{ \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \gamma \in \operatorname{Im} \mathbb{K} \setminus \{0\} \right\}.$$

Note that for $\mathbb{K} = \mathbb{R}$, the root spaces $\mathfrak{g}_{\pm 2\alpha}$ do not exist. Moreover, we have

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \psi & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & -\psi^* \end{pmatrix} : E \in \mathrm{Mat}(n-1, n-1; \mathbb{K}), E^* = -E, \psi \in \mathbb{K} \right\}.$$

It is now easy to see that $\mathfrak{a}$ is in fact maximal because $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. It follows that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda.$$

2.6 Hyperbolic spaces as semidirect products

2.6.1 Summary of the results

This section presents a view of hyperbolic spaces through a group of isometries and provides the translation of the Riemannian and metric space structures between the different descriptions.

This group of isometries is the connected subgroup NA of $O_{\mathbb{K}}(Q)$. It consists of products of elements of two further subgroups N and A, and N is precisely the group that we later associate with the visual boundary. Matrices in NA can be parameterised in terms of three components $(u, s, a) \in \mathbb{K}^{n-1} \times \mathrm{Im}\ \mathbb{K} \times \mathbb{R}$, which allows for a smooth identification between this product manifold and NA. Moreover, we demonstrate that NA acts simply transitively on $\mathbb{K}\mathbf{H}^n$. We set $\mathrm{Im}\ \mathbb{R} = \{0\}$ so that we do not need to treat this case separately.

Employing the group structure of NA, we define a multiplication law for $\mathbb{K}^{n-1} \times \mathrm{Im}\ \mathbb{K} \times \mathbb{R}$ which turns it into the semidirect product $(\mathbb{K}^{n-1} \ltimes \mathrm{Im}\ \mathbb{K}) \rtimes \mathbb{R}$. The simply transitive action of NA on $\mathbb{K}\mathbf{H}^n$ provides a smooth map between the hyperbolic spaces and NA, and therefore also between $\mathbb{K}\mathbf{H}^n$ and the semidirect product. With the aim of fully translating the Riemannian manifold structure between the different perspectives, we use general results about symmetric spaces to express the Riemannian metric and the distance function in terms of $(\mathbb{K}^{n-1} \ltimes \mathrm{Im}\ \mathbb{K}) \rtimes \mathbb{R}$. The main result of this section is the following theorem.

Theorem 2.21. *The manifold* $(\mathbb{K}^{n-1} \ltimes \mathrm{Im}\ \mathbb{K}) \rtimes \mathbb{R}$ *with the multiplication law*

$$(u, s, a) \cdot (v, t, b) = (u + e^a v, s + e^{2a} t + \mathrm{Im}\ (e^a u v^*), a + b),$$

is isomorphic to a matrix Lie group NA which is a subgroup of $O_{\mathbb{K}}(Q)_\circ$, and there is a smooth isometric correspondence of NA with $\mathbb{K}\mathbf{H}^n$. In terms of this semidirect product manifold, the left-invariant Riemannian metric looks as follows

$$g_{(0,0,0)}((u, s, a), (u, s, a)) = a^2 + \frac{|u|^2}{2} - \frac{s^2}{4},$$

and the left-invariant distance d is given by

$$4 \cosh^2 d(0, (v, t, b)) = 4 \cosh^2(b) + 2e^{-b} \cosh(b) \, |v|^2 + e^{-2b} \left(\frac{|v|^4}{4} + |t|^2 \right).$$

Note that the square of any nonzero $s \in \operatorname{Im} \mathbb{K}$ is a negative real number and that therefore the inner product $g_{(0,0,0)}$ is indeed positive definite.

The proof of Theorem 2.21 is divided into three parts, each addressing a specific aspect. The first part, treated in Section 2.6.2, focuses on defining and identifying the various manifolds. Here, we introduce the groups N and A and derive the multiplication law on $(\mathbb{K}^{n-1} \ltimes \operatorname{Im} \mathbb{K}) \rtimes \mathbb{R}$. Establishing the simple transitive action of the group NA on $\mathbb{K}\mathbf{H}^n$ leads to the identification of both NA and $(\mathbb{K}^{n-1} \ltimes \operatorname{Im} \mathbb{K}) \rtimes \mathbb{R}$ with $\mathbb{K}\mathbf{H}^n$. The second step involves translating the Riemannian metric to NA. It is discussed in Section 2.6.3. Finally, the last part of the proof, covered in Section 2.6.4, deals with expressing the distance function in the semidirect product representation. Given the parameterisation of NA by $\mathbb{K}^{n-1} \times \operatorname{Im} \mathbb{K} \times \mathbb{R}$, the translation between the two descriptions is straightforward.

2.6.2 Smooth identification of manifolds

In this section, we give the first part of the proof of Theorem 2.21. Specifically, we show that the hyperbolic spaces $\mathbb{K}\mathbf{H}^n$ can be associated with the semidirect product $(\mathbb{K}^{n-1} \ltimes \operatorname{Im} \mathbb{K}) \rtimes \mathbb{R}$, endowed with the multiplication law stated in Theorem 2.21. This semidirect product is introduced through the product of two subsets N and A of $O_{\mathbb{K}}(Q)$, which inherit a group structure from the group $O_{\mathbb{K}}(Q)$. We first define these sets and the set NA, which consists of products of elements from N and A, derive the group multiplication and inversion laws and use them to define the semidirect product. Furthermore, we show that the group action of NA on $\mathbb{K}\mathbf{H}^n$ is simply transitive. The last statement yields the smooth identification of $\mathbb{K}\mathbf{H}^n$ and NA as well as $\mathbb{K}\mathbf{H}^n$ and $(\mathbb{K}^{n-1} \ltimes \operatorname{Im} \mathbb{K}) \rtimes \mathbb{R}$. Our approach is based on Section 10.5 in [LD23].

Definition 2.22. We define the following subsets of $O_{\mathbb{K}}(Q)$.

- Let $a \in \mathbb{R}$ and define

$$A(a) := \begin{pmatrix} e^a & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & e^{-a} \end{pmatrix}.$$

  By A we denote the one-parameter set of matrices of this form, that is,

$$A = \{A(a) \colon a \in \mathbb{R}\}.$$

- Let $u^{\mathsf{T}} \in \mathbb{K}^{n-1}$ and $s \in \operatorname{Im} \mathbb{K}$. We define

$$h(u, s) := \begin{pmatrix} 1 & u & \frac{|u|^2}{2} + s \\ 0 & I_{n-1} & u^* \\ 0 & 0 & 1 \end{pmatrix}.$$

By N we denote the set of all such matrices, that is,

$$N = \left\{ h(u,s) \colon u^\mathsf{T} \in \mathbb{K}^{n-1}, s \in \operatorname{Im} \mathbb{K} \right\}.$$

- By NA we denote the set of products of matrices in N and A, that is,

$$NA = \left\{ h(u,s)A(a) \colon u^\mathsf{T} \in \mathbb{K}^{n-1}, s \in \operatorname{Im} \mathbb{K}, a \in \mathbb{R} \right\}.$$

Using Proposition 2.16, it is a simple calculation to check that N, A and NA are subsets of $O_{\mathbb{K}}(Q)$. For convenience, we drop the more accurate notation $u^\mathsf{T} \in \mathbb{K}^{n-1}$ when referring to the row vectors in the matrices of N from now on and simply write $u \in \mathbb{K}^{n-1}$.

The next lemma summarises some multiplication laws in N, A and NA that will be useful for the following proofs.

Lemma 2.23. *The multiplication of matrices in N and A satisfies the following.*

(a) For $u \in \mathbb{K}^{n-1}$, $s \in \operatorname{Im} \mathbb{K}$ and $a \in \mathbb{R}$ we have

$$A(a)h(u,s) = h(e^a u, e^{2a} s)A(a).$$

(b) For $u, v \in \mathbb{K}^{n-1}$ and $s, t \in \operatorname{Im} \mathbb{K}$ we have

$$h(u,s)h(v,t) = h(u+v, s+t+\operatorname{Im}(uv^*)).$$

(c) For $a, b \in \mathbb{R}$ we have

$$A(a)A(b) = A(a+b).$$

Proof. The lemma can be proven by carrying out the matrix computations.

To prove *(a)*, let $u \in \mathbb{K}^{n-1}$, $s \in \operatorname{Im} \mathbb{K}$ and $a \in \mathbb{R}$. We compute the matrix product

$$\begin{aligned}
A(a)h(u,s) &= \begin{pmatrix} e^a & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & e^{-a} \end{pmatrix} \begin{pmatrix} 1 & u & \frac{|u|^2}{2} + s \\ 0 & I_{n-1} & u^* \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^a & e^a u & e^a \left(\frac{|u|^2}{2} + s \right) \\ 0 & I_{n-1} & u^* \\ 0 & 0 & e^{-a} \end{pmatrix} \\
&= \begin{pmatrix} 1 & e^a u & e^{2a} \left(\frac{|u|^2}{2} + s \right) \\ 0 & I_{n-1} & e^a u^* \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^a & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & e^{-a} \end{pmatrix} = h(e^a u, e^{2a} s)A(a).
\end{aligned}$$

For the proof of *(b)*, let $u, v \in \mathbb{K}^{n-1}$ and $s, t \in \mathrm{Im}\ \mathbb{K}$. We determine the multiplication law

$$\begin{aligned} h(u,s)h(v,t) &= \begin{pmatrix} 1 & u & \frac{|u|^2}{2}+s \\ 0 & I_{n-1} & u^* \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v & \frac{|v|^2}{2}+t \\ 0 & I_{n-1} & v^* \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & u+v & \frac{|u|^2+|v|^2}{2}+uv^*+s+t \\ 0 & I_{n-1} & u^*+v^* \\ 0 & 0 & 1 \end{pmatrix} = h(u+v, s+t+\mathrm{Im}\ (uv^*)), \end{aligned}$$

where we used the identity $|u+v|^2 = |u|^2 + |v|^2 + 2\mathrm{Re}\ (uv^*)$ to rewrite

$$\frac{|u|^2+|v|^2}{2} + uv^* = \frac{|u+v|^2}{2} + \mathrm{Im}\ (uv^*).$$

Moreover, for the proof of *(c)*, let $a, b \in \mathbb{R}$. Then

$$\begin{aligned} A(a)A(b) &= \begin{pmatrix} e^a & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & e^{-a} \end{pmatrix} \begin{pmatrix} e^b & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & e^{-b} \end{pmatrix} \\ &= \begin{pmatrix} e^{a+b} & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & e^{-(a+b)} \end{pmatrix} = A(a+b). \end{aligned}$$

□

Equipped with these multiplications laws, we characterise the group structure of NA.

Lemma 2.24. *We obtain the following multiplication and inversion laws for elements in NA.*

(a) For $u, v \in \mathbb{K}^{n-1}, s, t \in \mathrm{Im}\ \mathbb{K}$ and $a, b \in \mathbb{R}$ it holds that

$$h(u,s)A(a)h(v,t)A(b) = h(u + e^a v, s + e^{2a} t + \mathrm{Im}\ (e^a uv^*))A(a+b).$$

(b) For $u \in \mathbb{K}^{n-1}, s \in \mathrm{Im}\ \mathbb{K}, a \in \mathbb{R}$ it holds that

$$(h(u,s)A(a))^{-1} = h(-e^{-a}u, -e^{-2a}s)A(-a).$$

Proof. For the proof of *(a)*, let $u, v \in \mathbb{K}^{n-1}, s, t \in \mathrm{Im}\ \mathbb{K}$ and $a, b \in \mathbb{R}$. Using the results from Lemma 2.23, we obtain

$$\begin{aligned} h(u,s)A(a)h(v,t)A(b) &= h(u,s)h(e^a v, e^{2a} t)A(a)A(b) \\ &= h(u + e^a v, s + e^{2a} t + \mathrm{Im}\ (e^a uv^*))A(a+b). \end{aligned}$$

Finally, to prove *(b)*, we again use the previously derived multiplication laws to see that

$$h(-e^{-a}u, -e^{-2a}s)A(-a)h(u,s)A(a) = I_{n+1}.$$

The statement follows from the properties of the matrix inverse. □

We derive some important statements about N, A and NA.

Lemma 2.25. *The following hold.*

(a) A and N are subgroups of $O_{\mathbb{K}}(Q)_{\circ}$.

(b) NA is a subgroup of $O_{\mathbb{K}}(Q)_{\circ}$.

(c) N is normal in NA.

Proof. To prove *(a)*, note that $I_{n+1} = A(0) = h(0,0)$ is an element of both A and N. The sets are closed under multiplication, this follows from Lemmas 2.23. To see that they are also closed under inversion, we apply part *(b)* of Lemma 2.24 while setting $a = 0$ or $u = 0$ and $s = 0$ respectively. From the definition of N, A and NA it is clear that they are connected.
The statement *(b)* now immediately follows from Lemma 2.24.
To prove *(c)*, let $u, v \in \mathbb{K}^{n-1}, s, t \in \text{Im } \mathbb{K}$ and $a \in \mathbb{R}$. With Lemma 2.24 we see that

$$\begin{aligned} &h(u,s)A(a)h(v,t)(h(u,s)A(a))^{-1} \\ &\qquad = h(u,s)A(a)h(v,t)h(-e^{-a}u, -e^{-2a}s)A(-a) \\ &\qquad = h(u,s)h(e^{a}(v - e^{-a}u), e^{2a}(t - e^{-2a}s + \text{Im } (-e^{-a}vu^{*})))A(a)A(-a) \\ &\qquad = h(u,s)h(e^{a}v - u, e^{2a}t - s + e^{2a}\text{Im } (-e^{-a}vu^{*}))), \end{aligned}$$

and from Lemma 2.23 we know that this is in N. □

Using the group structure of NA and its obvious identification with $\mathbb{K}^{n-1} \times \text{Im } \mathbb{K} \times \mathbb{R}$, we arrive at the following definition.

Definition 2.26. By $(\mathbb{K}^{n-1} \ltimes \text{Im } \mathbb{K}) \rtimes \mathbb{R}$ we denote the manifold $\mathbb{K}^{n-1} \times \text{Im } \mathbb{K} \times \mathbb{R}$ equipped with the multiplication law

$$(u,s,a) \cdot (v,t,b) = (u + e^{a}v, s + e^{2a}t + \text{Im } (e^{a}uv^{*}), a + b).$$

Per construction, we can identify NA with the semidirect product $(\mathbb{K}^{n-1} \ltimes \text{Im } \mathbb{K}) \rtimes \mathbb{R}$.

Corollary 2.27. *The map*

$$\begin{array}{rccc} \varphi\colon & (\mathbb{K}^{n-1} \ltimes \text{Im } \mathbb{K}) \rtimes \mathbb{R} & \to & NA \\ & (u,s,a) & \mapsto & h(u,s)A(a) \end{array}$$

is an isomorphism of Lie groups.

Moreover, we can show that NA acts simply transitively on the hyperbolic space $\mathbb{K}\mathbf{H}^n$. This allows us to derive the identification of the manifolds stated in Theorem 2.21.

Proposition 2.28. *The subgroup NA acts simply transitively on $\mathbb{K}\mathbf{H}^n$.*

Proof. Let $\hat{o} = (1, 0, \cdots, 0, 1)$ and $o = [\hat{o}]$. A simple calculation shows that

$$h(u,s)A(a).o = \begin{bmatrix} e^{2a} + \frac{|u|^2}{2} + s \\ u^* \\ 1 \end{bmatrix}.$$

Let $[x] \in \mathbb{K}\mathbf{H}^n$ with representative $x = (x_1, \cdots, x_n, 1)$. Then, the equation $h(u,s)A(a).\hat{o} = x$ is equivalent to the system of equations $x_1 = e^{2a} + \frac{|u|^2}{2} + s$ and $(x_2, \cdots, x_n) = \bar{u}$, so we set $u = (\overline{x}_2, \cdots, \overline{x}_n)$. Moreover, we can choose $s = \text{Im } x_1$ and $a = \frac{1}{2}\log(\text{Re } x_1 - \frac{|u|^2}{2})$. Note that the last equation must have a solution because $[x] \in \mathbb{K}\mathbf{H}^n$ implies that $\text{Re } x_1 - \frac{|u|^2}{2} = -\langle x \,|\, x\rangle$ is positive. If $\mathbb{K} = \mathbb{R}$, we have $s = 0$, which is consistent with $\mathbb{R}$ not having an imaginary part. It is important that the solution of these equations are unique because we fixed a lift of $[x]$ by setting the last entry in homogeneous coordinates to 1.

For general $[y] \in \mathbb{K}\mathbf{H}^n$, we now know that there exists a unique $g_y \in NA$ such that $g_y.o = [y]$. With Lemma 2.24 we obtain its inverse g_y^{-1}. Moreover, for each $[x] \in \mathbb{K}\mathbf{H}^n$ there is a unique g_x with $g_x.o = [x]$. We can thus write $g_x g_y^{-1}.[y] = [x]$. □

Together with Corollary 2.27, this proposition provides the wanted identification and therefore proves the first part of Theorem 2.21.

Corollary 2.29. *The map* $(u,s,a) \mapsto h(u,s)A(a).o$ *gives a smooth identification between* $\mathbb{K}\mathbf{H}^n$ *and* $(\mathbb{K}^{n-1} \ltimes \text{Im } \mathbb{K}) \rtimes \mathbb{R}$ *as manifolds.*

2.6.3 The Riemannian metric

Without explicitly stating it, we have proven the first part of the following special case of a more general result from the theory of symmetric spaces (see for example [Hel78] and the introduction to our Section 2.5). We point out that due to the fact that the group action is simply transitive, the point stabiliser of o is trivial and therefore, the familiar quotient ${}^{NA}\!/_{\text{Stab}_{NA}(o)}$ can be identified with NA.

Fact 2.30. The orbit map

$$\text{orb}_o\colon NA \to \mathbb{K}\mathbf{H}^n,\ g \mapsto g.o$$

provides a smooth correspondence between the manifolds NA and $\mathbb{K}\mathbf{H}^n$. Moreover, given that $\text{orb}_o(e) = o$, its differential is a isomorphism of tangent spaces,

$$d_e\text{orb}_o\colon T_eNA \to T_o\mathbb{K}\mathbf{H}^n, \tag{2.5}$$

where $e = h(0,0)A(0)$ is the identity element of NA.

With these isomorphisms, we can explicitly derive an expression for the Riemannian metric g on $\mathbb{K}\mathbf{H}^n$ in terms of NA. This is the next step in proving Theorem 2.21.

For $v, w \in T_eNA$, we set

$$g_e^{(NA)}(v,w) = g_o(d_e\mathrm{orb}_o(v), d_e\mathrm{orb}_o(w)). \tag{2.6}$$

It is sufficient to determine the metric of the Lie group NA at the identity, as we require it to be left-invariant. Hence, g_e can be translated to other points using left multiplication in NA. Let $u \in \mathbb{K}^{n-1}$, $s \in \mathrm{Im}\ \mathbb{K}$ and $a \in \mathbb{R}$, then

$$L_{h(u,s)A(a)}(e) = h(u,s)A(a),$$

and the map $L_{h(u,s)A(a)}$ is a diffeomorphism (with smooth inverse $L_{(h(u,s)A(a))^{-1}}$). Therefore,

$$d_eL_{h(u,s)A(a)} \colon T_eNA \to T_{h(u,s)A(a)}NA$$

is an isomorphism. Then, the metric on $T_{h(u,s)A(a)}NA$ can be pulled back to the metric on T_eNA,

$$g^{(NA)}_{h(u,s)A(a)} = L^*_{h(u,s)A(a)}g_e^{(NA)}.$$

The tangent space T_eNA is of course the Lie algebra of NA. In the next lemma, we determine its elements.

Lemma 2.31. *The Lie algebra of NA can be written as a direct sum of vector spaces,*

$$\mathrm{Lie}(NA) = \mathfrak{n} \oplus \mathfrak{a},$$

where $\mathfrak{n}$ contains matrices of the form

$$v(u,s) = \begin{pmatrix} 0 & u & s \\ 0 & 0 & u^* \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{where } u \in \mathbb{K}^{n-1} \text{ and } s \in \mathrm{Im}\ \mathbb{K},$$

and $\mathfrak{a}$ contains matrices of the form

$$w(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix}, \qquad \text{where } a \in \mathbb{R}.$$

Proof. This follows directly from the definitions of N and A. □

We proceed to understanding the orbit map and its differential. It is useful to choose a representative $\hat{o}$ of o and first determine the orbit map $\mathrm{orb}_{\hat{o}} \colon NA \to \mathbb{K}^{n,1}$ and its differential, and then project the result into the hyperbolic space $\mathbb{K}\mathbf{H}^n$ respectively its tangent space, using the fact that the following diagrams commute,

$$\begin{array}{cccc} & NA & \xrightarrow{\mathrm{orb}_{\hat{o}}} & \mathbb{K}^{n,1} \\ id_{NA} & \downarrow & & \downarrow \;\pi \\ & NA & \xrightarrow{\mathrm{orb}_{o}} & \mathbb{K}\mathbf{H}^n \end{array}$$

and

$$\begin{array}{cccc} & T_eNA & \xrightarrow{d_e\mathrm{orb}_{\hat{o}}} & T_{\hat{o}}\mathbb{K}^{n,1} \\ id_{T_eNA} & \downarrow & & \downarrow \;d_{\hat{o}}\pi \\ & T_eNA & \xrightarrow{d_e\mathrm{orb}_{o}} & T_o\mathbb{K}\mathbf{H}^n \end{array}.$$

We choose the representative $\hat{o} = (1, 0, \cdots, 0, 1)$ and determine $\mathrm{orb}_{\hat{o}}$ and $d_e\mathrm{orb}_{\hat{o}}$ in the following lemma.

Lemma 2.32. *For $h(u,s)A(a) \in NA$, where $u \in \mathbb{K}^{n-1}, s \in \mathrm{Im}\ \mathbb{K}$ and $a \in \mathbb{R}$, it is*

$$\mathrm{orb}_{\hat{o}}(h(u,s)A(a)) = \begin{pmatrix} e^a + e^{-a}\left(\frac{|u|^2}{2} + s\right) \\ e^{-a}u^* \\ e^{-a} \end{pmatrix},$$

and for $v(u,s) + w(a) \in \mathrm{Lie}(NA)$, where $u \in \mathbb{K}^{n-1}, s \in \mathrm{Im}\ \mathbb{K}$ and $a \in \mathbb{R}$, we have

$$d_e\mathrm{orb}_{\hat{o}}(v(u,s) + w(a)) = \begin{pmatrix} a+s \\ u^* \\ -a \end{pmatrix}.$$

Proof. These are straightforward calculations. □

To obtain $d_e\mathrm{orb}_o$ from the results of Lemma 2.32, we need an observation about the differential of the projection.

Lemma 2.33. *The kernel of $d_{\hat{o}}\pi$ is the following set,*

$$\ker d_{\hat{o}}\pi = \mathrm{span}_{\mathbb{K}}\{\hat{o}\}.$$

Proof. We prove the lemma for the case $\mathbb{K} = \mathbb{R}$ only here and defer the general proof to Section C of the appendix. The other two cases are not fundamentally different, but there are more real coordinates, leading to lengthier calculations and expressions. In fact, the general proof shows that the cases $\mathbb{K} = \mathbb{C}, \mathbb{H}$ can be thought of as replacing the real numbers in the proof below with complex numbers respectively quaternions.

First, we note that for any $[(x_1, \cdots, x_{n+1})] \in \mathbb{R}\mathbf{H}^n$ it holds that

$$\langle x \,|\, x \rangle = \sum_{i=2}^{n} {x_1}^2 - 2x_1x_{n+1} < 0,$$

and therefore $x_{n+1} \neq 0$. This allows us to choose coordinates on $\mathbb{R}\mathbf{H}^n$ by setting

$$\begin{array}{cccc} \varphi\colon & \mathbb{R}\mathbf{H}^n & \to & \mathbb{R}^n \\ & [(x_1, \cdots, x_{n+1})] & \mapsto & \left(x_1 x_{n+1}^{-1}, \cdots, x_n x_{n+1}^{-1}\right). \end{array}$$

This is independent of the choice of a representative because when replacing the representative x of $[x]$ by a scalar multiple $x\lambda$ for some $\lambda \in \mathbb{R} \setminus \{0\}$, the scalar cancels in $\varphi([x\lambda])$. In terms of these coordinates, the projection π can be expressed as

$$\varphi(\pi(x_1, \cdots, x_{n+1})) = \left(x_1 x_{n+1}^{-1}, \cdots, x_n x_{n+1}^{-1}\right).$$

Its differential at $\hat{o}$ can now be computed explicitly, it is

$$d_{\hat{o}}(\varphi \circ \pi) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

and we can read off its kernel,

$$\ker d_{\hat{o}}\pi = \left\{(\lambda, 0, \cdots, 0, \lambda)^{\mathsf{T}} : \lambda \in \mathbb{R}\right\}. \qquad \square$$

Obtaining both orb_o and $d_e\mathrm{orb}_o$ is now simple.

Proposition 2.34. *For $h(u, s)A(a) \in NA$, where $u \in \mathbb{K}^{n-1}, s \in \mathrm{Im}\ \mathbb{K}$ and $a \in \mathbb{R}$, it is*

$$\mathrm{orb}_o(h(u, s)A(a)) = \begin{bmatrix} e^a + e^{-a}\left(\frac{|u|^2}{2} + s\right) \\ e^{-a}u^* \\ e^{-a} \end{bmatrix},$$

and for $v(u, s) + w(a) \in \mathrm{Lie}(NA)$, where $u \in \mathbb{K}^{n-1}, s \in \mathrm{Im}\ \mathbb{K}$ and $a \in \mathbb{R}$, the tangent vector $d_e\mathrm{orb}_o(v(u, s) + w(a)) \in T_o\mathbb{K}\mathbf{H}^n$ in the model where we identify $T_o\mathbb{K}\mathbf{H}^n$ with $\hat{o}^\perp$ is represented by $(a + \frac{s}{2}, u^*, -a - \frac{s}{s})$. We also write this as

$$d_e\mathrm{orb}_o(v(u, s) + w(a)) = \begin{bmatrix} a + \frac{s}{2} \\ u^* \\ -a - \frac{s}{s} \end{bmatrix}.$$

Proof. The statement about orb_o is clear. To see that the statement about the differential of the orbit map is true, we recall from Lemma 2.32 that

$$d_e\mathrm{orb}_{\hat{o}}(v(u, s) + w(a)) = \begin{pmatrix} a + s \\ u^* \\ -a \end{pmatrix},$$

and we can use Lemma 2.33 to find the representative of the tangent vector $d_{\hat{o}}\pi(a+s, u^*, -a)$ in our model of the tangent space $T_o\mathbb{K}\mathbf{H}^n$ as $\hat{o}^\perp$, which is

$$d_e\mathrm{orb}_o(v(u,s)+w(a)) = \begin{bmatrix} a+\frac{s}{2} \\ u^* \\ -a-\frac{s}{s} \end{bmatrix}. \qquad \square$$

To express the Riemannian metric in terms of NA using (2.6), we only need to derive $g_e^{(NA)}$ from g_o. We therefore determine g_o.

Lemma 2.35. *Using the identification of $T_o\mathbb{K}\mathbf{H}^n$ with $\hat{o}^\perp$, it holds for any two tangent vectors $U, V \in T_o\mathbb{K}\mathbf{H}^n$ represented by $u, v \in \hat{o}^\perp$, where $u = (u_1, \cdots, u_{n+1})$ and $v = (v_1, \cdots, v_{n+1})$, that*

$$g_o(U,V) = \overline{u}_1 v_1 + \frac{1}{2}\sum_{i=2}^{n} \overline{u}_i v_i.$$

Proof. First note that

$$(1,0,\cdots,0,1)^\perp = \left\{(y_1,\cdots,y_{n+1}) \in \mathbb{K}^{n+1} \colon y_{n+1} = -y_1\right\}.$$

This implies that $u_{n+1} = -u_1$ and $v_{n+1} = -v_1$. Inserting this into the definition of the Riemannian metric (2.2), we find

$$\begin{aligned} g_o(U,V) = -\frac{\langle u \mid v\rangle}{\langle \hat{o} \mid \hat{o}\rangle} &= \frac{1}{2}\left(\sum_{i=2}^{n}\overline{u}_i v_i - \overline{u}_1 v_{n+1} - \overline{u}_{n+1} v_1\right) \\ &= \frac{1}{2}\left(\sum_{i=2}^{n}\overline{u}_i v_i + \overline{u}_1 v_1 + \overline{u}_1 v_1\right) \\ &= \overline{u}_1 v_1 + \frac{1}{2}\sum_{i=2}^{n}\overline{u}_i v_i. \end{aligned} \qquad \square$$

With all these pieces, we can express the Riemannian metric in terms of NA. Note that due to the polarisation identity, it is sufficient to determine the inner product $g_e^{(NA)}$ only for identical arguments.

Proposition 2.36. *Let $u \in \mathbb{K}^{n-1}, s \in \operatorname{Im}\mathbb{K}$ and $a \in A$. Then*

$$g_e^{(NA)}(v(u,s)+w(a), v(u,s)+w(a)) = a^2 + \frac{|u|^2}{2} - \frac{s^2}{4}.$$

Proof. Using (2.6) and the results from Proposition 2.34 and Lemma 2.35, we find

$$\begin{aligned} g_e^{(NA)}(v(u,s)+w(a), v(u,s)+w(a)) &= g_o(d_e\pi_o(v(u,s)+w(a)), d_e\pi_o(v(u,s)+w(a))) \\ &= \frac{1}{2}\Big\langle (a+\frac{s}{2}, u^*, -a-\frac{s}{2} \Big| a+\frac{s}{2}, u^*, -a-\frac{s}{2}) \Big\rangle \\ &= a^2 + \frac{|u|^2}{2} - \frac{s^2}{4}. \end{aligned} \qquad \square$$

Note that this is indeed positive definite because for $s \in \mathrm{Im}\ \mathbb{K}$ with $s = s_1 i + s_2 j + s_3 k$ (where $s_1, s_2, s_3 \in \mathbb{R}$ and we take $s_2 = s_3 = 0$ if $\mathbb{K} = \mathbb{C}$ and $s_1 = s_2 = s_3 = 0$ if $\mathbb{K} = \mathbb{R}$), it is $-s^2 = s_1^2 + s_2^2 + s_3^2$.

With the identification of NA with $(\mathbb{K}^{n-1} \ltimes \mathrm{Im}\ \mathbb{K}) \rtimes \mathbb{R}$ from Corollary 2.27, we have therefore proven the second part of Theorem 2.21.

Before proceeding to translating the distance function to NA and the semidirect product, we add a further note about the Riemannian metric on NA. For calculations, another inner product on the Lie algebra $\mathrm{Lie}(NA)$ is often convenient, which is the trace form

$$(X, Y) \mapsto \mathrm{Re}\ \mathrm{tr}(X^* Y).$$

In the following, we show that, up to irrelevant scalar prefactors, the trace form defines the same left-invariant metric as the Riemannian metric on $\mathbb{K}\mathbf{H}^n$ after identifying the tangent spaces via (2.5). We begin by calculating the trace form on $\mathrm{Lie}(NA)$.

Lemma 2.37. *The trace form defines a positive definite inner product on $T_e NA$, which is, for $u \in \mathbb{K}^{n-1}$, $s \in \mathrm{Im}\ \mathbb{K}$ and $a \in \mathbb{R}$, given by*

$$(v(u,s) + w(a), v(u,s) + w(a)) \mapsto 2a^2 + 2\,|u|^2 - s^2.$$

Proof. Let $u \in \mathbb{K}^{n-1}, s \in \mathrm{Im}\ \mathbb{K}$ and $a \in \mathbb{R}$, then

$$\mathrm{Re}\ \mathrm{tr}((v(u,s)+w(a))^*(v(u,s)+w(a))) = \mathrm{Re}\ \mathrm{tr} \begin{pmatrix} a & u & s \\ 0 & 0 & u^* \\ 0 & 0 & -a \end{pmatrix}^* \begin{pmatrix} a & u & s \\ 0 & 0 & u^* \\ 0 & 0 & -a \end{pmatrix} = 2a^2 + 2\,|u|^2 - s^2.$$

□

Comparing the results of Proposition 2.36 and Lemma 2.37 yields the following corollary.

Corollary 2.38. *The inner product resulting from the trace form corresponds, up to rescaling, to the Riemannian metric on NA.*

2.6.4 The distance function

The goal of this section is to express the distance function in terms of NA. We follow Section 10.5 in [LD23]. Using Fact 2.30, we know that the distance function on NA is related to the distance function d on $\mathbb{K}\mathbf{H}^n$ by

$$d^{(NA)}(h(u,s)A(a), h(v,t)A(b)) = d(\mathrm{orb}_o(h(u,s)A(a)), \mathrm{orb}_o(h(v,t)A(b))). \tag{2.7}$$

Given that we defined d using the quadratic form Q, we first need to express $\langle \cdot \mid \cdot \rangle$ in terms of the parameters of NA. Since this is done through representatives, we specify a way to lift $\mathrm{orb}_o(h(u,s)A(a))$ into $\mathbb{K}^{n,1}$. As in the previous sections, we let $\hat{o}$ be the representative of o with $\hat{o} = (1, 0, \cdots, 0, 1)$.

Due to the fact that the group action of NA is simply transitive, we can identify all $[x] \in \mathbb{K}\mathbf{H}^n$ with a unique element $h(u,s)A(a) \in NA$ by choosing $u \in \mathbb{K}^{n-1}, s \in \operatorname{Im} \mathbb{K}$ and $a \in \mathbb{R}$ such that $h(u,s)A(a).o = [x]$. Further, we define $\hat{x} \in \mathbb{K}^{n,1}$ to be the representative of $[x]$ that is given by $\operatorname{orb}_{\hat{o}}(h(u,s)A(a))$. We know that the definition of the distance function (2.1) is independent of the choice of a representative so that we can evaluate it for representatives of the form $\hat{x}$.

We start with three lemmas about the evaluation of $\langle \cdot \,|\, \cdot \rangle$.

Lemma 2.39. *Let $[x] \in \mathbb{K}\mathbf{H}^n$, then $\langle \hat{x} \,|\, \hat{x} \rangle = -2$.*

Proof. From Proposition 2.20 we know that there exists an element $g_x \in NA$ such that $g_x.o = [x]$. The element g_x is an isometry which implies that

$$\langle \hat{x} \,|\, \hat{x} \rangle = \langle g_x.\hat{o} \,|\, g_x.\hat{o} \rangle = \langle \hat{o} \,|\, \hat{o} \rangle = -2. \qquad \square$$

Our next goal is to find $\langle \hat{x} \,|\, \hat{y} \rangle$ for general $\hat{x}, \hat{y} \in \mathbb{K}\mathbf{H}^n$. To do so, we first determine $\langle \hat{o} \,|\, \hat{y} \rangle$.

Lemma 2.40. *Let $[y] \in \mathbb{K}\mathbf{H}^n$ and $h(v,t)A(b) \in NA$ be such that $h(v,t)A(b).o = [y]$. Then*

$$|\langle \hat{o} \,|\, \hat{y} \rangle|^2 = 4\cosh^2(b) + 2e^{-b}\cosh(b)\,|v|^2 + e^{-2b}\left(\frac{|v|^4}{4} + |t|^2\right).$$

Proof. We prove the lemma by explicitly writing $\langle \hat{o} \,|\, \hat{y} \rangle$. To do so, we determine $\hat{y}$,

$$\hat{y} = h(v,t)A(b).\hat{o} = \begin{pmatrix} e^b + e^{-b}\left(\frac{|v|^2}{2} + t\right) \\ e^{-b}v^* \\ e^{-b} \end{pmatrix}.$$

The calculation of $\langle \hat{o} \,|\, \hat{y} \rangle$ is now straightforward. We find

$$\langle \hat{o} \,|\, \hat{y} \rangle = \hat{o}^{\mathsf{T}} K \hat{y} = -2\cosh(b) - e^{-b}\left(\frac{|v|^2}{2} + t\right).$$

Squaring the norm of the result yields

$$|\langle \hat{o} \,|\, \hat{y} \rangle|^2 = 4\cosh^2(b) + 2e^{-b}\cosh(b)\,|v|^2 + e^{-2b}\left(\frac{|v|^4}{4} + |t|^2\right). \qquad \square$$

With the previous lemma, we can determine $\langle \hat{x} \,|\, \hat{y} \rangle$ for any $[x], [y] \in \mathbb{K}\mathbf{H}^n$.

Lemma 2.41. *For $[x], [y] \in \mathbb{K}\mathbf{H}^n$ let $h(u,s)A(a)$ and $h(v,t)A(b)$ be the unique elements in NA such that $h(u,s)A(a).o = [x]$ and $h(v,t)A(b).o = [y]$. Then*

$$|\langle \hat{x} \,|\, \hat{y} \rangle|^2 = 4\cosh^2(b-a) + 2e^{-a-b}\cosh(b-a)\,|v-u|^2 + e^{-2(a+b)}\left(\frac{|v-u|^4}{4} + |t - s - \operatorname{Im}(uv^*)|^2\right).$$

Proof. To prove the lemma, we make use of the fact that $\langle \cdot \,|\, \cdot \rangle$ is invariant under multiplication of both arguments by elements of NA. Moreover, we use the group structure of NA as given in

Lemma 2.24 to trace the general case back to the expression computed in Lemma 2.40. This yields

$$\begin{aligned}
\langle \hat{x} \,|\, \hat{y} \rangle &= \langle h(u,s)A(a).\hat{o} \,|\, h(v,t)A(b).\hat{o} \rangle \\
&= \langle \hat{o} \,|\, (h(u,s)A(a))^{-1} h(v,t)A(b).\hat{o} \rangle \\
&= \langle \hat{o} \,|\, h(-e^{-a}u, -e^{-2a}s)A(-a)h(v,t)A(b).\hat{o} \rangle \\
&= \langle \hat{o} \,|\, h(e^{-a}(v-u), e^{-2a}(t-s-\mathrm{Im}\,(uv^*)))A(b-a).\hat{o} \rangle \\
&= -2\cosh(b-a) - e^{-(b-a)} \left(\frac{e^{-2a}\,|v-u|^2}{2} + e^{-2a}(t-s-\mathrm{Im}\,(uv^*)) \right).
\end{aligned}$$

After squaring the norm of this result, we arrive at

$$|\langle \hat{x} \,|\, \hat{y} \rangle|^2 = 4\cosh^2(b-a) + 2e^{-a-b}\cosh(b-a)\,|v-u|^2 + e^{-2(a+b)} \left(\frac{|v-u|^4}{4} + |t-s-\mathrm{Im}\,(uv^*)|^2 \right).$$

□

Equipped with these lemmas, we are ready to express the distance function in terms of NA and the semidirect product. We denote both distance functions as $d^{(NA)}$ because the identification of NA with $(\mathbb{K}^{n-1} \ltimes \mathrm{Im}\,\mathbb{K}) \rtimes \mathbb{R}$ is straightforward.

Proposition 2.42. *With the identification of NA with $\mathbb{K}\mathbf{H}^n$ from Fact 2.30 and the identification of $(\mathbb{K}^{n-1} \ltimes \mathrm{Im}\,\mathbb{K}) \rtimes \mathbb{R}$ with NA from Corollary 2.27, the distance on $(\mathbb{K}^{n-1} \ltimes \mathrm{Im}\,\mathbb{K}) \rtimes \mathbb{R}$ is given by*

$$4\cosh^2 d^{(NA)}(0,(v,t,b)) = 4\cosh^2(b) + 2e^{-b}\cosh(b)\,|v|^2 + e^{-2b}\left(\frac{|v|^4}{4} + |t|^2\right).$$

It is left-invariant with respect to the product structure of $(\mathbb{K}^{n-1} \ltimes \mathrm{Im}\,\mathbb{K}) \rtimes \mathbb{R}$.

Proof. The left-invariance of the distance function follows from the definition of the distance on $\mathbb{K}\mathbf{H}^n$ and the fact that the action of $(\mathbb{K}^{n-1} \ltimes \mathrm{Im}\,\mathbb{K}) \rtimes \mathbb{R}$ on itself is by isometries. It is therefore sufficient to determine the distance of any element $(v,t,b) \in (\mathbb{K}^{n-1} \ltimes \mathrm{Im}\,\mathbb{K}) \rtimes \mathbb{R}$ to the identity $(0,0,0)$. This is achieved by using (2.7) and inserting the results of Lemmas 2.39 and 2.40 into the definition of the distance function (2.1). The result of this procedure is easily seen to be

$$4\cosh^2 d^{(NA)}(0,(v,t,b)) = 4\cosh^2(b) + 2e^{-b}\cosh(b)\,|v|^2 + e^{-2b}\left(\frac{|v|^4}{4} + |t|^2\right).$$ □

This proposition finishes the proof of Theorem 2.21.

For further reference, we derive an expression for the distance of any two points (u,s,a) and (v,t,b) in $(\mathbb{K}^{n-1} \ltimes \mathrm{Im}\,\mathbb{K}) \rtimes \mathbb{R}$.

Corollary 2.43. *The distance of* $(u,s,a),(v,t,b) \in (\mathbb{K}^{n-1} \ltimes \mathrm{Im}\ \mathbb{K}) \rtimes \mathbb{R}$ *is given by*

$$4\cosh^2 d^{(NA)}((u,s,a),(v,t,b)) =$$
$$4\cosh^2(b-a) + 2e^{-a-b}\cosh(b-a)\,|v-u|^2 + e^{-2(a+b)}\left(\frac{|v-u|^4}{4} + |t-s-\mathrm{Im}\ (uv^*)|^2\right).$$

3 The visual boundaries of the $\mathbb{K}$-hyperbolic spaces

3.1 The visual boundary

The purpose of this section is, besides introducing the concept of the visual boundary, to prove that three objects, the group N from Definition 2.22, the visual boundary minus one point, referred to as pt, and any horosphere centred at pt, are identified with each other. We first define the visual boundary as the set of equivalence classes of geodesic rays, where two geodesic rays are equivalent whenever they are asymptotic. Subsequently, we prove that for the space $\mathbb{K}\mathbf{H}^n$ we can associate its visual boundary $\partial\mathbb{K}\mathbf{H}^n$ (minus $\{\mathrm{pt}\}$) with N. We further show that the orbits of N are the horospheres centred at pt. This section follows Chapter II.10 in [BH99] and Section 11 in [LD23]. Identifying these three objects is essential to defining a subRiemannian manifold structure on the visual boundary, as each provides a different perspective from which a given property or concept is particularly well established.

The identification of the visual boundary $\partial\mathbb{K}\mathbf{H}^n$ (minus one point) with a subgroup of $O_{\mathbb{K}}(Q)_\circ$ is not canonical, but it depends on the choice of the extra point in the visual boundary. All possible choices of such a point are equivalent because $O_{\mathbb{K}}(Q)_\circ$ acts transitively on $\partial\mathbb{K}\mathbf{H}^n$. This transitive action enables us to translate one such subgroup into another. We can therefore choose for the extra point our favourite point in the visual boundary, which is the point $\mathrm{pt} = [(1,0,\cdots,0)]$. Within the previously derived setup, this is our favourite point because it yields the identification of $\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ with N.

We begin by defining the visual boundary. Recall from Section 2.3 that a unit speed geodesic ray $c\colon [0,\infty) \to \mathbb{K}\mathbf{H}^n$ is a curve that satisfies

$$d(c(t),c(t')) = |t-t'| \quad \text{for all } t,t' \in [0,\infty),$$

and that two geodesic rays c and c' are asymptotic if there exists a constant C_0 such that

$$d(c(t),c'(t)) \le C_0 \quad \text{for all } t \ge 0.$$

In Lemma 2.12 we proved that we can obtain an equivalence relation on the set of unit speed geodesic rays by defining asymptotic geodesic rays to be equivalent. We use this equivalence relation now to define the visual boundary.

Definition 3.1. We introduce an equivalence relation on the set of geodesic rays by declaring two geodesic rays equivalent if and only if they are asymptotic. We define the *visual boundary* $\partial\mathbb{K}\mathbf{H}^n$ as the set of equivalence classes of unit speed geodesic rays.

Since we modelled the hyperbolic space $\mathbb{K}\mathbf{H}^n$ as a subspace of $\mathbb{K}\mathbf{P}^n$, there is a way to realise its boundary as a subset of $\mathbb{K}\mathbf{P}^n$. This perspective is helpful to prove that we can identify N with $\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$.

Proposition 3.2. *We can identify the visual boundary $\partial\mathbb{K}\mathbf{H}^n$ with the set of nullvectors in $\mathbb{K}\mathbf{P}^n$,*

$$\partial\mathbb{K}\mathbf{H}^n = \{[x] \in \mathbb{K}\mathbf{P}^n \colon \langle x \,|\, x\rangle = 0\}\,.$$

Proof. From Lemma 2.14 we know that two geodesic rays

$$\begin{aligned} c(t) &= \left[x \cosh\left(\sqrt{\langle u \,|\, u\rangle} t\right) + \frac{u}{\sqrt{\langle u \,|\, u\rangle}} \sinh\left(\sqrt{\langle u \,|\, u\rangle} t\right)\right], \\ c'(t) &= \left[y \cosh\left(\sqrt{\langle v \,|\, v\rangle} t\right) + \frac{v}{\sqrt{\langle v \,|\, v\rangle}} \sinh\left(\sqrt{\langle v \,|\, v\rangle} t\right)\right], \end{aligned}$$

where we take $[x], [y] \in \mathbb{K}\mathbf{H}^n$ and $u, v \in x^\perp, y^\perp$ respectively, are asymptotic if and only if $\left[x + \frac{u}{\sqrt{\langle u|u\rangle}}\right] = \left[y + \frac{v}{\sqrt{\langle v|v\rangle}}\right]$. We note that

$$\left[x \cosh\left(\sqrt{\langle u \,|\, u\rangle} t\right) + u \sinh\left(\sqrt{\langle u \,|\, u\rangle} t\right)\right] = \left[x + \frac{u}{\sqrt{\langle u \,|\, u\rangle}} \tanh\left(\sqrt{\langle u \,|\, u\rangle} t\right)\right]$$

converges to $\left[x + \frac{u}{\sqrt{\langle u|u\rangle}}\right]$ as $t \to \infty$, where the limit is taken with respect to the quotient topology of $\mathbb{K}\mathbf{P}^n$. It is easy to see that $\left[x + \frac{u}{\sqrt{\langle u|u\rangle}}\right]$ is a nullvector. We assume again $\langle x \,|\, x\rangle = -1$, otherwise we replace x by $\frac{x}{\sqrt{|\langle x|x\rangle|}}$ and u by $\frac{u}{\sqrt{|\langle x|x\rangle|}}$ respectively to obtain

$$\left\langle x + \frac{u}{\sqrt{\langle u \,|\, u\rangle}} \,\middle|\, x + \frac{u}{\sqrt{\langle u \,|\, u\rangle}} \right\rangle = \langle x \,|\, x\rangle + \frac{\langle u \,|\, u\rangle}{\langle u \,|\, u\rangle} = -1 + 1 = 0.$$

This concludes the proof that we can identify every equivalence class with such a nullvector.

It remains to show that the other implication also holds, that is, given a nullvector $[w] \in \mathbb{K}\mathbf{P}^n$, there exists an equivalence class of geodesic rays that can be associated with it. We therefore choose any $x \in \mathbb{K}\mathbf{H}^n$ with $\langle x \,|\, x\rangle = -1$. It must hold that $\langle x \,|\, w\rangle \neq 0$ because the restriction of $\langle \cdot \,|\, \cdot\rangle$ to $x^\perp$ is positive definite, so that no $w \in x^\perp$ with $w \neq 0$ could satisfy $\langle w \,|\, w\rangle = 0$. Then it is true that $-w\langle x \,|\, w\rangle^{-1} - x \in x^\perp$, because

$$\langle x \,|\, w\langle x \,|\, w\rangle^{-1} + x\rangle = \langle x \,|\, x\rangle + \langle x \,|\, w\rangle\langle x \,|\, w\rangle^{-1} = -1 + 1 = 0,$$

and that $\langle w\langle x \,|\, w\rangle^{-1} + x \,|\, w\langle x \,|\, w\rangle^{-1} + x\rangle = 1$, because

$$\langle w\langle x \,|\, w\rangle^{-1} + x \,|\, w\langle x \,|\, w\rangle^{-1} + x\rangle = \langle x \,|\, x\rangle + 2\mathrm{Re}\ \langle x \,|\, w\rangle\langle x \,|\, w\rangle^{-1} = -1 + 2 = 1.$$

Thus, we can define a geodesic ray c by setting

$$c(t) = \left[x \cosh t + \left(-w\langle x \,|\, w\rangle^{-1} - x\right) \sinh t\right],$$

and the calculation above shows that the limit point of c in $\mathbb{K}\mathbf{P}^n$ as $t \to \infty$ is

$$\left[x + \left(-w\langle x \,|\, w\rangle^{-1} - x\right)\right] = \left[-w\langle x \,|\, w\rangle^{-1}\right] = [w]. \qquad \square$$

We can therefore characterise an equivalence class of geodesic rays uniquely by the nullvector in $\mathbb{K}\mathbf{P}^n$ to which its elements converge in the topology of $\mathbb{K}\mathbf{P}^n$ as $t \to \infty$. This explains why we sometimes refer to the elements of the visual boundary as the *'points at infinity'*.

The identification of $\partial\mathbb{K}\mathbf{H}^n$ minus one point with a subgroup of $O_{\mathbb{K}}(Q)_\circ$ depends on the choice of a point in the visual boundary. We choose the point $\mathrm{pt} = [(1, 0, \cdots, 0)]$ and define a geodesic ray c_0 by

$$c_0(t) = [(e^t, 0, \cdots, 0, e^{-t})]. \tag{3.1}$$

Then pt is the limit point of $c_0(t)$ as $t \to \infty$ with respect to the quotient topology of $\mathbb{K}\mathbf{P}^n$. Moreover, A acts by transvections on c_0, that is,

$$A(t').c_0(t) = c_0(t + t') \quad \text{for all } t, t' \in \mathbb{R}.$$

We need the following two lemmas.

Lemma 3.3. *For each $h \in N$ it holds that*

$$\lim_{t\to\infty} A(-t)hA(t) = I_{n+1}.$$

Proof. Let $h \in N$, then h is of the form

$$h = \begin{pmatrix} 1 & u & \frac{|u|^2}{2} + s \\ 0 & I_{n-1} & u^* \\ 0 & 0 & 1 \end{pmatrix},$$

where $u \in \mathbb{K}^{n-1}$ and $s \in \mathrm{Im}\,\mathbb{K}$. A simple matrix calculation shows that

$$A(-t)hA(t) = \begin{pmatrix} 1 & e^{-t}u & e^{-2t}\left(\frac{|u|^2}{2} + s\right) \\ 0 & I_{n-1} & e^{-t}u^* \\ 0 & 0 & 1 \end{pmatrix},$$

which converges to the unit matrix as $t \to \infty$. $\square$

Lemma 3.4. *Let $c_0 \colon \mathbb{R} \to \mathbb{K}\mathbf{H}^n$ be defined by* (3.1)*. The following statements hold.*

(a) N fixes $\lim_{t\to\infty} c_0(t)$.

(b) *The geodesic rays issuing from points in $N.o$ that are asymptotic to c_0 are precisely the geodesics $t \mapsto h.c_0(t)$, where $h \in N$.*

Proof. Given that A acts on c_0 by transvections, we can write $c_0(t) = A(t).c_0(0)$. We make use of this identity for the proof of both parts of the lemma.

To prove *(a)*, we write $\lim_{t\to\infty} c_0(t) = \lim_{t\to\infty} A(t).c_0(0)$. Then, using the left-invariance of the distance, it is evident that for any $h \in N$,

$$d(hA(t).c_0(0), A(t).c_0(0)) = d(A(-t)hA(t).c_0(0), c_0(0)),$$

which tends to zero as $t \to \infty$, because from Lemma 3.3 we know that $A(-t)hA(t) \to I_{n+1}$ as $t \to \infty$.

For the proof of *(b)*, consider the map $t \mapsto h.c_0(t)$ for $h = h(u,s) \in N$, where $u \in \mathbb{K}^{n-1}$ and $s \in \operatorname{Im} \mathbb{K}$. Using the left-invariant distance function $d^{(NA)}$ from Proposition 2.42, we can write the distance of two points $h.c_0(t)$ and $h.c_0(t')$ along the curve as

$$\begin{aligned} 4\cosh^2 d(h.c_0(t), h.c_0(t')) &= 4\cosh^2 d^{(NA)}((u,s,t'),(u,s,t)) \\ &= 4\cosh^2 d^{(NA)}((0,0,t'),(0,0,t)) \\ &= 4\cosh^2(t-t'), \end{aligned}$$

so that $d(h.c_0(t), h.c_0(t')) = |t-t'|$ follows. We conclude that $t \mapsto h.c_0(t)$ is a geodesic ray.

For proving that this geodesic ray is asymptotic to c_0, we determine the distance of $c_0(t)$ and $h.c_0(t)$,

$$d(h.c_0(t), c_0(t)) = d(hA(t).c_0(0), A(t).c_0(0)) = d(A(-t)hA(t).c_0(0), c_0(0)).$$

To show that this vanishes as $t \to \infty$, we write $h = h(u,s)$ for $u \in \mathbb{K}^{n-1}$ and $s \in \operatorname{Im} \mathbb{K}$ and recall from Lemma 2.24 that

$$A(t)h(u,s) = h(e^t u, e^{2t} s)A(t).$$

Then $A(-t)h(u,s)A(t) = h(e^{-t}u, e^{-2t}s)A(-t)A(t) = h(e^{-t}u, e^{-2t}s)$, so that with the distance function $d^{(NA)}$, we have

$$4\cosh^2 d^{(NA)}((0,0,0),(e^{-t}u, e^{-2t}s, 0)) = 4 + 2e^{-2t}|u|^2 + e^{-4t}\left(\frac{|u|^4}{4} + |s|^2\right) \qquad \text{for all } t \in [0,\infty).$$

Note that $(0,0,0)$ corresponds to the point $c_0(0)$ and $(e^{-t}u, e^{-2t}s, 0)$ corresponds to $A(-t)hA(t).c_0(0)$. The exponentials involving t are clearly bounded by 1 for $t \in [0,\infty)$, so that

$$d(c_0(t), h.c_0(t)) \le \left(\operatorname{arcosh}\left(1 + \frac{1}{2}|u|^2 + \frac{1}{4}\left(\frac{|u|^4}{4} + |s|^2\right)\right)\right)^{1/2} \qquad \text{for all } t \in [0,\infty).$$

It follows that the geodesic rays c_0 and $h.c_0$ are asymptotic.

To see that these are all the geodesic rays issuing from points in $N.o$ that are asymptotic to c_0, recall from Lemma 2.14 that given an initial point $h.o$ in $N.o$, there is precisely one asymptotic geodesic ray c that issues from $h.o$ and is asymptotic to c_0. This geodesic ray, however, is already known to be $t \mapsto h.c_0(t)$. □

Note that any geodesic ray c whose image is equal to the image of c_0 (up to some geodesic line segment of finite length) but does not necessarily have the same initial point at time $t = 0$ is asymptotic to c_0 because, as a consequence of the non-arbitrariness of its parameterisation, it is related to c_0 by a transvection $A(t_c)$ for some constant $t_c \in \mathbb{R}$,

$$c(t) = A(t_c).c_0(t) \quad \text{for all } t \in [0, \infty),$$

which implies that

$$d(c(t), c_0(t)) = d(c(t), A(t_c).c_0(t)) = |t_c| \,.$$

This is constant and hence bounded for all $t \in \mathbb{R}$, and it follows that c_0 and c are asymptotic. Therefore, part *(b)* of the previous lemma is sufficient to characterise the elements of the equivalence class of c_0.

To prove that N can be identified with $\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ we define the following map. Let $\zeta_0 \in N.o$ and $u \in \mathbb{K}^{n-1}, s \in \operatorname{Im} \mathbb{K}$ such that $h(u, s)$ is the unique element in N with $h(u, s).o = \zeta_0$ (its existence and uniqueness follow from Proposition 2.20). Define the map

$$\zeta \colon \mathbb{R} \to \mathbb{K}\mathbf{H}^n, \quad t \mapsto \zeta_t = h(u, s)A(-t).o. \tag{3.2}$$

The following lemma summarises the relevant properties of this map and allows us to extend it to $\mathbb{R} \cup \{-\infty, +\infty\}$.

Lemma 3.5. *For all $\zeta_0 \in N.o$, the following are true.*

(a) $\zeta|_{[0,\infty)}$ *is a geodesic ray.*

(b) $\lim\limits_{t\to-\infty} \zeta_t = \mathrm{pt}$.

(c) $\zeta_\infty := \lim\limits_{t\to\infty} \zeta_t \in \partial\mathbb{K}\mathbf{H}^n$.

Proof. Note that $A(-t).o = c_0(-t)$. Then the statement *(a)* follows from a similar calculation as in the proof of Lemma 3.4 *(b)*, but is not already implied by the lemma because it only applies to $\zeta|_{(-\infty,0]}$. Using the left-invariance of $d^{(NA)}$, we can write the distance of two points ζ_t and $\zeta_{t'}$ along the curve as

$$\begin{aligned} 4\cosh^2 d(\zeta_t, \zeta_{t'}) &= 4\cosh^2 d^{(NA)}((u, s, -t'), (u, s, -t)) \\ &= 4\cosh^2 d^{(NA)}((0, 0, -t'), (0, 0, -t)) \\ &= 4\cosh^2(t' - t), \end{aligned}$$

so that $d(\zeta_t, \zeta_{t'}) = |t' - t|$ follows.

The second claim is proven in Lemma 3.4 *(a)*.

To prove *(c)*, we explicitly calculate ζ_∞. This is

$$\lim_{t\to\infty}\zeta_t = \lim_{t\to\infty}\begin{bmatrix} e^{-t}+e^{t}\left(\frac{|u|^2}{2}+s\right) \\ e^{t}u^* \\ e^{t}\end{bmatrix} = \lim_{t\to\infty}\begin{bmatrix} e^{-2t}+\frac{|u|^2}{2}+s \\ u^* \\ 1\end{bmatrix} = \begin{bmatrix} \frac{|u|^2}{2}+s \\ u^* \\ 1\end{bmatrix}. \tag{3.3}$$

Then $\left\langle (\frac{|u|^2}{2}+s, u^*, 1) \middle| (\frac{|u|^2}{2}+s, u^*, 1)\right\rangle = 0$ and with the identification $\partial\mathbb{K}\mathbf{H}^n = \{[x]\in\mathbb{K}\mathbf{P}^n : \langle x\,|\,x\rangle = 0\}$ it follows that $\zeta_\infty \in \partial\mathbb{K}\mathbf{H}^n$. □

The maps ζ yield the identification of $\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ and $N.o$.

Proposition 3.6. *The set $N.o$ can be identified with $\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$.*

Proof. Let $\varphi\colon N.o \to \partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ such that $\varphi(\zeta_0) = \zeta_\infty$. The injectivity of the map follows from the explicit form of ζ_∞ that was determined in (3.3). It remains to prove the surjectivity of φ. We choose $x \in \mathbb{K}^{n,1}$ such that $[x] \in \partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ and $x = (x_1, \cdots, x_n, 1)$. We set $u = (\overline{x}_2, \cdots, \overline{x}_n)$, and since $\langle x\,|\,x\rangle = 0$ implies that $2\mathrm{Re}\ x_1 = \sum_{i=2}^{n} |x_i|^2$, we can set $s = \mathrm{Im}\ x_1$. We therefore have $\varphi(h(u,s).o) = [x]$. □

At this point, it becomes apparent why we have to exclude the point pt from the visual boundary $\partial\mathbb{K}\mathbf{H}^n$ to identify it with $N.o$. The identification is achieved through the map $\varphi(h.o) = \lim_{t\to\infty} hA(-t).o = h.\left(\lim_{t\to\infty} A(-t).o\right)$. The fact that that the action of N on $\partial\mathbb{K}\mathbf{H}^n$ fixes pt ensures that no other point of the visual boundary can be mapped to pt by the action of a group element of N, because otherwise we could act on pt with the inverse of this element and would obtain $\lim_{t\to\infty}(A(-t)).o$, which is a point in the visual boundary but clearly not pt.

We proceed by deriving the identification of the group N with the horospheres centred at pt. Before proving this identification, we define horospheres in $\mathbb{K}\mathbf{H}^n$.

Definition 3.7. The spheres about $[x_k] \in \mathbb{K}\mathbf{H}^n$ are the level sets of the functions

$$\rho_k([x]) = \frac{\langle x\,|\,x_k\rangle\langle x_k\,|\,x\rangle}{\langle x\,|\,x\rangle}.$$

For $[x_k] \to [y] \in \partial\mathbb{K}\mathbf{H}^n$, the functions ρ_k converge and the *horospheres centred at* $[y]$ have the form

$$H_{r,[y]} = \{[x]\in\mathbb{K}\mathbf{H}^n : r\langle x\,|\,x\rangle = \langle x\,|\,y\rangle\langle y\,|\,x\rangle\},$$

where $r < 0$.

Now we can prove that the orbits of N are the horospheres centred at pt.

Proposition 3.8. *The orbits of N are the horospheres centred at the point* pt.

Proof. Recall from Lemma 3.4 that N fixes the point pt. Let $r < 0$ and $[x] \in H_{r,\mathrm{pt}}$, then we have $\frac{\langle x|\mathrm{pt}\rangle\langle \mathrm{pt}|x\rangle}{\langle x|x\rangle} = r$. We write $\mathrm{pt} = \lim_{t\to\infty} A(t).c_0(0)$ and choose $x_0 \in \mathbb{K}^{n,1}$ such that $[x_0] = c_0(0)$. Using the definition of the distance function (2.1), it is

$$\begin{aligned} r = \lim_{t\to\infty} \frac{\langle x \mid A(t).x_0\rangle\langle A(t).x_0 \mid x\rangle}{\langle x \mid x\rangle} &= \lim_{t\to\infty} \langle A(t).x_0 \mid A(t).x_0\rangle d([x], A(t).c_0(0)) \\ &= \lim_{t\to\infty} \langle x_0 \mid x_0\rangle d([x], A(t).c_0(0)), \end{aligned}$$

where the last equality holds because $A(t)$ is an isometry. The prefactor $\langle x_0 \mid x_0\rangle$ is irrelevant, and the action of any $h \in N$ leaves invariant the distance $d([x], A(t).c_0(0))$ in the limit $t \to \infty$, because

$$\begin{aligned} \lim_{t\to\infty} \Big| d(h.[x], A(t).c_0(0)) - d([x], A(t).c_0(0)) \Big| &= \lim_{t\to\infty} \Big| d([x], h^{-1}A(t).c_0(0)) - d([x], A(t).c_0(0)) \Big| \\ &\le \lim_{t\to\infty} d(c_0(0), A(-t)h^{-1}A(t).c_0(0)) = 0, \end{aligned}$$

where we used the reverse triangle inequality in the second step. Hence, $[x] \in H_{r,\mathrm{pt}}$ if and only if $h.[x] \in H_{r,\mathrm{pt}}$ □

The desired identification of N, the visual boundary (minus $\{\mathrm{pt}\}$) and the horospheres centred at pt now follows.

Theorem 3.9. *N can be identified with $\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ and with any horosphere centred at* pt.

Proof. We know from Section 2.6 that the orbit map $\mathrm{orb}_o \colon N \to N.o,\ h \mapsto h.o$ is an isometric isomorphism. Then Propositions 3.6 and 3.8 immediately imply the claim of the theorem. □

3.2 Bijectivity of the exponential map

In the last section, we saw that the group N is identified with $\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$, and in the following, we always choose the description with N when making our considerations explicit. This identification yields convenient coordinates for the visual boundary (minus $\{\mathrm{pt}\}$) because we can define coordinates for N using a basis of its Lie algebra. We explain the definition of these coordinates in detail in Section 4. The reason why this is possible is that the exponential map is a diffeomorphism. Proving that this is true is the purpose of this section.

Proposition 3.10. *The exponential map* $\exp\colon \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \to N$ *is a diffeomorphism.*

Proof. We begin by revisiting our knowledge of the elements of N and its Lie algebra. Per definition, N contains matrices of the form

$$h(u,s) = \begin{pmatrix} 1 & u & \frac{|u|^2}{2} + s \\ 0 & I_{n-1} & u^* \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{where } u \in \mathbb{K}^{n-1} \text{ and } s \in \operatorname{Im} \mathbb{K},$$

and we know from Lemma 2.31 that the Lie algebra of N is

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & u & s \\ 0 & 0 & u^* \\ 0 & 0 & 0 \end{pmatrix} : u \in \mathbb{K}^{n-1}, s \in \operatorname{Im} \mathbb{K} \right\},$$

which is clearly $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$.

The exponential map is always smooth, and for proving that it is a diffeomorphism, we present its inverse. To do so, we first compute for $b_\alpha \in \mathfrak{g}_\alpha$ and $b_{2\alpha} \in \mathfrak{g}_{2\alpha}$ the exponential of $b_\alpha + b_{2\alpha}$, because this makes it easy to determine the inverse of exp. The element b_α can be written as

$$b_\alpha = \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & u^* \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{where } u \in \mathbb{K}^{n-1},$$

and $b_{2\alpha}$ can be written as

$$b_{2\alpha} = \begin{pmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{where } s \in \operatorname{Im} \mathbb{K}.$$

It is easy to see that

$$\exp(b_\alpha + b_{2\alpha}) = \begin{pmatrix} 1 & u & \frac{|u|^2}{2} + s \\ 0 & I_{n-1} & u^* \\ 0 & 0 & 1 \end{pmatrix},$$

and that this is precisely the form in which we defined the matrices in N. Thus determining the preimage of an element in N under the exponential map is straightforward. We define

$$\begin{array}{rrcl} F\colon & N & \to & \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \\ & h(u,s) = & \mapsto & \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & u^* \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{array}$$

The map F is obviously smooth and the inverse of the exponential map, and it follows that exp is a diffeomorphism. □

3.3 The visual boundary as a subRiemannian manifold

3.3.1 subRiemannian manifolds

In this section, we establish that for $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$, the visual boundary $\partial\mathbb{K}\mathbf{H}^n$ minus one point can be equipped with some structure that makes it a subRiemannian manifold. This also holds when $\mathbb{K} = \mathbb{R}$, but in that case, the visual boundary $\partial\mathbb{R}\mathbf{H}^n$ minus one point is isometrically isomorphic to $\mathbb{R}^{n-1}$ and therefore a Riemannian manifold (which is trivially a subRiemannian manifold). The metric on the visual boundary depends on the choice of the extra point, and even given such a choice, there are still many different metrics possible, but these are in a sense equivalent, as discussed below. We choose $\mathrm{pt} = [(1, 0, \cdots, 0)]$ for the extra point because we know from Theorem 3.9 that $\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ is identified with N, but we emphasise that this choice, although convenient, is arbitrary and does not make any conceptual difference.

In the first part of this section we introduce the concept of a subRiemannian manifold, following Section 0.3 in [LD23], and in the second part, we explain the structure that makes the visual boundary $\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ a subRiemannian manifold. We explicitly establish this for N, and with Theorem 3.9 our result can easily be translated to other descriptions of the visual boundary.

Definition 3.11. A distribution Δ is *bracket-generating* if every tangent vector $v \in TM$ is a linear combination of $X_1, [X_2, X_3], [X_4, [X_5, X_6]], \cdots$, where $X_1, X_2, \cdots$ are tangent to Δ.

Definition 3.12. A *subRiemannian manifold* is a triple (M, Δ, g), where M is a differentiable manifold, Δ is a bracket generating distribution and g is a smooth section of positive definite quadratic forms on Δ.

Definition 3.13. If (M, Δ, g) is a subRiemannian manifold, then a curve γ in M is *horizontal* if it is piecewise smooth and $\dot\gamma(t) \in \Delta_{\gamma(t)}$ for all t. We call Δ_p, where $p \in M$, the *horizontal subspace* of T_pM.

In our case, g can be obtained by restricting a Riemannian metric on M to Δ. This allows for the definition of vertical subspaces.

Definition 3.14. Let (M, g) be a Riemannian manifold with a distribution Δ such that $(M, \Delta, g|_\Delta)$ is a subRiemannian manifold. Let $p \in M$ and Δ_p the horizontal subspace of T_pM. Then its orthogonal complement $\Delta_p^\perp$ (with respect to g_p) is the *vertical subspace* of the tangent space T_pM. A curve γ in M is *vertical* if it is piecewise smooth and $\dot\gamma(t) \in \Delta^\perp_{\gamma(t)}$ for all t.

Definition 3.15. For a subRiemannian manifold (M, Δ, g), we can define a distance function, the *subRiemannian distance*, by

$$d(p, q) = \inf \{\mathrm{Length}(\gamma) \colon \gamma \text{ horizontal from } p \text{ to } q\}.$$

This definition makes sense because the condition that Δ is bracket-generating guarantees that for any pair of points p, q, there is always a horizontal curve connecting them (for a proof, see Theorem 3.1.17 in [LD23]).

3.3.2 The subRiemannian structure of the visual boundaries

In this section, we explain how to obtain a distribution Δ and a Riemannian metric g on the group N from Definition 2.22 so that $(N, \Delta, g|_\Delta)$ is a subRiemannian manifold. Recall from Section 3.1 that the orbits of N are the horospheres centred at pt, and due to the fact that NA acts simply transitively on $\mathbb{K}\mathbf{H}^n$, we can identify each horosphere with N. Moreover, we can identify N with

$\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$. This naturally (but not canonically!) provides us with a left-invariant metric on the visual boundary by restricting the Riemannian metric on $\mathbb{K}\mathbf{H}^n$ to the horosphere. Since the horospheres are embedded submanifolds of $\mathbb{K}\mathbf{H}^n$, it is clear that the restriction of the Riemannian metric on $\mathbb{K}\mathbf{H}^n$ is again a metric. We also know from Corollary 2.38 that this metric can be identified with the left-invariant metric on N defined by the trace form. Of course, the metric depends on the choice of horosphere, but since any two left-invariant metrics on a Lie group are bi-Lipschitz equivalent [LD23, Proposition 5.4.1], the choice does not matter, because lengths and areas obtained from different metrics differ by uniformly bounded amounts. We thus set

$$g_e(X, Y) = \mathrm{Re}\ \mathrm{tr}(X^*Y) \tag{3.4}$$

for $X, Y \in T_eN$, and translate this to other points by left-multiplication.

Moreover, the fact that $\mathrm{Lie}(N) = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ enables the definition of a distribution Δ by declaring $\mathfrak{g}_\alpha$ horizontal at the identity and translating this to other points through left-multiplication. Since we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = \mathfrak{g}_{2\alpha}$, this distribution is clearly bracket-generating. Formally, we define Δ by

$$\Delta_{h(u,s)} = d_e L_{h(u,s)} \mathfrak{g}_\alpha. \tag{3.5}$$

The metric and the distribution give N (and thus the visual boundary minus $\{\mathrm{pt}\}$) the structure of a subRiemannian manifold. We state this important result as a theorem.

Theorem 3.16. *Let N be the group associated with the visual boundary $\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ of the $\mathbb{K}$-hyperbolic space (minus the point* pt*). Let g be the left-invariant metric defined by* (3.4) *and Δ be the distribution defined by* (3.5)*. Then $(N, \Delta, g|_\Delta)$ is a subRiemannian manifold.*

Obviously, this is less interesting when $\mathbb{K} = \mathbb{R}$ than in the other cases, because in the real case, the root space $\mathfrak{g}_{2\alpha}$ does not exist, hence the whole tangent space is horizontal and the subRiemannian distance coincides with the distance induced by the Riemannian metric, so that $\partial\mathbb{R}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ is not only a subRiemannian, but a Riemannian manifold. In the complex and the quaternionic case, however, there are one- respectively three-dimensional vertical subspaces and these are what makes the geometry on the visual boundary exciting.

We point out that the groups N corresponding to $\partial\mathbb{K}\mathbf{H}^n \setminus \{\mathrm{pt}\}$ are precisely the $\mathbb{K}$-Heisenberg groups $\mathbb{K}\mathcal{H}^n$ [LD23]. In the next part of the thesis, we denote the groups N as $\mathbb{K}\mathcal{H}^n$ for two reasons. Firstly, we will specifically choose $\mathbb{K}$ (and sometimes also n), which can be made more clear in the notation $\mathbb{K}\mathcal{H}^n$. The second reason is more practical, it is simply to keep the notation in line with the notation of our main reference.

4 Having fun with the subRiemannian manifold

4.1 The case of $\mathbb{C}\mathcal{H}^{n+1}$

In order to present an application of the established concepts, we dedicate the final section of the thesis to solving a geometric problem on the visual boundary. Our goal is to generalise the following proposition, which applies to paths in $\mathbb{C}\mathcal{H}^{n+1}$ and is taken from a paper by Allcock [All98], to special vertical paths in $\mathbb{H}\mathcal{H}^{n+1}$.

Proposition. *[All98, Lemma 4.3] Any regular vertical path in $\mathbb{C}\mathcal{H}^{n+1}$ of length L is homotopic to a horizontal path of length $2\sqrt{\pi L}$ by a homotopy of area at most $2L + \frac{4\sqrt{\pi}}{3}L^{3/2}$.*

We use the term 'path' with the meaning of 'smooth curve' here.

Allcock uses the proposition as a lemma to prove an isoperimetric inequality for the groups $\mathbb{C}\mathcal{H}^{n+1}$. An elaboration of the full context is beyond the scope of this thesis and we simply treat his lemma as a statement that is interesting for the purpose of working with a subRiemannian manifold. In the first part of this section, we show Allcock's proof of the proposition. To do so, we establish the subRiemannian structure of $\mathbb{C}\mathcal{H}^{n+1}$ and in particular the horizontal and vertical subspaces in detail by choosing coordinates and expressing the distribution in terms of these coordinates explicitly.

For choosing coordinates for $\mathbb{C}\mathcal{H}^{n+1}$, recall from Proposition 3.10 that the exponential map is a diffeomorphism. We can therefore choose a basis of the Lie algebra and define the coordinates on $\mathbb{C}\mathcal{H}^{n+1}$ through this basis, where the exponential map provides the necessary unique correspondence of elements of $\mathbb{C}\mathcal{H}^{n+1}$ to the Lie algebra elements. For $b = (b_1, \cdots, b_n) \in \mathbb{C}^n$ and $s \in \operatorname{Im} \mathbb{C}$ we set

$$P_b := \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & b^* \\ 0 & 0 & 0 \end{pmatrix},$$
$$M_s := \begin{pmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that

$$\mathfrak{g}_\alpha = \{P_b \colon b \in \mathbb{C}^n\},$$
$$\mathfrak{g}_{2\alpha} = \{M_s \colon s \in \operatorname{Im} \mathbb{C}\}.$$

Note that $b \in \mathbb{C}^n$ now because $\mathbb{C}\mathcal{H}^{n+1}$ corresponds to the visual boundary of $\mathbb{K}\mathbf{H}^{n+1}$ (minus $\{\text{pt}\}$).

There are $2n+1$ real basis vectors in the Lie algebra. We choose the basis vectors

$$X_j = P_{e_j},$$
$$Y_j = P_{ie_j},$$
$$Z = M_i,$$

where e_j denotes the j^{th} standard basis vector in $\mathbb{R}^n$. The only nonvanishing brackets are

$$[X_j, Y_j] = -2Z \quad \text{for all } j = 1, \ldots, n.$$

Since we have a basis of the Lie algebra of $\mathbb{C}\mathcal{H}^{n+1}$ that identifies its elements with elements of

$\mathbb{C}^n \times \operatorname{Im} \mathbb{C}$, which again is obviously identified with $\mathbb{R}^{2n+1}$, we can choose the following coordinates on $\mathbb{C}\mathcal{H}^{n+1}$,

$$\begin{aligned}\varphi\colon & \mathbb{C}\mathcal{H}^{n+1} \to \mathbb{R}^{2n+1} \\ & \exp(P_{(x_1+iy_1,\cdots,x_n+iy_n)} + M_{iz}) \mapsto (x_1,\cdots,x_n,y_1,\ldots,y_n,z),\end{aligned}$$

with parameterisation

$$\begin{aligned}\psi\colon & \mathbb{R}^{2n+1} \to \mathbb{C}\mathcal{H}^{n+1} \\ & (x_1,\cdots,x_n,y_1,\ldots,y_n,z) \mapsto \exp(P_{(x_1+iy_1,\cdots,x_n+iy_n)} + M_{iz}).\end{aligned}$$

Due to the definition of the coordinates through the exponential map, we call these the *exponential coordinates* on $\mathbb{C}\mathcal{H}^{n+1}$.

Recall from Theorem 3.16 that an inner product on $T_e\mathbb{C}\mathcal{H}^{n+1}$ can be declared using the trace form. The basis vectors $X_1,\cdots,X_n,Y_1,\cdots,Y_n,Z$ are orthogonal with respect to this inner product, and rescaling the inner product makes them orthonormal. We take the rescaled inner product to be g_e. A metric on $T_p\mathbb{C}\mathcal{H}^{n+1}$ can be defined by translating this inner product to other points p by left multiplication L_p,

$$g_p = L_p^* g_e.$$

The choice of metric is not canonical, there are many different left-invariant metrics that we could choose instead, but they are all bi-Lipschitz equivalent to the one obtained from the trace form so that statements about lengths and areas would differ by at most a uniformly bounded amount. This might result in different prefactors in the statement of Proposition 4.3, but each metric yields the same power laws in L.

We identify the horizontal and vertical subspaces next. Following Theorem 3.16, the horizontal subspace at the identity H_e is given by $\mathfrak{g}_\alpha$, which is

$$H_e = \operatorname{span}_{\mathbb{R}}\{X_1,\cdots,X_n,Y_1,\cdots,Y_n\},$$

and it can be translated to other points $p \in \mathbb{C}\mathcal{H}^{n+1}$ by left-multiplication,

$$H_p = d_eL_pH_e = \operatorname{span}_{\mathbb{R}}\{d_eL_pX_1,\cdots,d_eL_pX_n,d_eL_pY_1,\cdots,d_eL_pY_n\}.$$

At each point $p \in \mathbb{C}\mathcal{H}^{n+1}$, the vertical subspace is the orthogonal complement $V_p = H_p^\perp$. From our choice of basis vectors it is clear that

$$V_p = \operatorname{span}_{\mathbb{R}}\{d_eL_pZ\} \quad \text{for all } p \in \mathbb{C}\mathcal{H}^{n+1}.$$

To prove the proposition, it is sufficient to provide all expressions in $\mathbb{C}\mathcal{H}^2$, because once we have the desired homotopy that takes a vertical path to a horizontal path in $\mathbb{C}\mathcal{H}^2$, we can use this horizontal path to define a horizontal path in $\mathbb{C}\mathcal{H}^{n+1}$ for any other $n \in \mathbb{N}$ by setting the remaining coordinates to zero. It is easy to see that this yields a horizontal path of the same length and does not change the area of the homotopy.

We therefore restrict the following discussion to $\mathbb{C}\mathcal{H}^2$. In terms of exponential coordinates, left multiplication becomes

$$(\varphi \circ L_{\psi(x,y,z)} \circ \psi)(a, b, c) = (x + a, y + b, z + c + (xb - ay)),$$

with differential

$$d_{\varphi(e)}(\varphi \circ L_{\psi(x,y,z)} \circ \psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y & x & 1 \end{pmatrix}.$$

The horizontal subspaces can be described as the kernels of a 1-form,

$$H_p = \ker \xi_p,$$

where we defined

$$\xi = \varphi^* \eta, \qquad \text{with} \qquad \eta_{(x,y,z)} = dz - (xdy - ydx)\,.$$

A curve $\beta\colon I \to \mathbb{C}\mathcal{H}^2$ is horizontal if and only if $\beta'(t) \in \ker \xi_{\beta(t)}$ for all $t \in I$, where β' denotes (here and in the rest of this section) the derivative of β by the curve parameter.

The representation of the horizontal subspaces as kernels of the 1-form ξ yields a geometric interpretation for the z-coordinate in the case of $\mathbb{C}\mathcal{H}^2$. We use Stokes' theorem to see that for the projection onto the first two components of the horizontal path in coordinates $\sigma = \pi(\varphi \circ \beta)$, we have

$$\int_\sigma dz = \int_\sigma (xdy - ydx) = \int_{D_\sigma} d(xdy - ydx) = 2 \int_{D_\sigma} dx \wedge dy = 2\text{Area}(D_\sigma),$$

where D_σ is the area in $\mathbb{R}^2$ which is bounded by σ and line segments from the origin to the start- and endpoint of σ. This makes sense because the integral $\int_\lambda (xdy - ydx)$ vanishes whenever λ is a radial line segment so that we can always add it to σ to obtain a closed loop which bounds D_σ. For a horizontal path in coordinates, the z-coordinate therefore measures twice the area that its projection onto $\mathbb{R}^2$ encloses (potentially after closing it with radial line segments).

Our considerations yield a way of lifting paths in $\mathbb{R}^2$ to horizontal paths in $\mathbb{C}\mathcal{H}^2$.

Definition 4.1. Let α be a path in $\mathbb{R}^2$ and $\pi\colon \mathbb{C}\mathcal{H}^2 \to \mathbb{R}^2, \psi(x, y, z) \mapsto (x, y)$. Then any horizontal curve $\tilde{\alpha}$ in $\mathbb{C}\mathcal{H}^2$ that satisfies the condition

$$\pi(\tilde{\alpha}) = \alpha$$

is a *horizontal lift* of α.

The existence of horizontal lifts is guaranteed by the next lemma.

Lemma 4.2. *For any path α in $\mathbb{R}^2$ there exists a horizontal curve $\tilde{\alpha}$ such that $\pi(\tilde{\alpha}) = \alpha$. If we specify one point through which the curve passes at some time t_0, this curve it unique.*

Proof. Let $I \subset \mathbb{R}$ be an interval and $\alpha\colon I \to \mathbb{R}^2$ be a smooth curve. We denote

$$\alpha(t) = \begin{pmatrix} \alpha_x(t) \\ \alpha_y(t) \end{pmatrix}.$$

Our previous considerations show that a horizontal curve $\tilde{\alpha}$ with $\pi(\tilde{\alpha}) = \alpha$ of the form

$$\varphi(\tilde{\alpha}(t)) = \begin{pmatrix} \tilde{\alpha}_x(t) \\ \tilde{\alpha}_y(t) \\ \tilde{\alpha}_z(t) \end{pmatrix}$$

must satisfy $\tilde{\alpha}_x = \alpha_x$ and $\tilde{\alpha}_y = \alpha_y$, and an equation determining the third component is obtained from the condition that the curve $\tilde{\alpha}$ is horizontal, which yields

$$\tilde{\alpha}_z' = \alpha_x \alpha_y' - \alpha_y \alpha_x'. \tag{4.1}$$

The equation (4.1) is a simple first-order differential equation, and it is solved by

$$\tilde{\alpha}_z(t) - \tilde{\alpha}_z(t_0) = \int_{t_0}^{t} \alpha_x(s)\alpha_y'(s) - \alpha_y(s)\alpha_x'(s)ds. \tag{4.2}$$

Recall that, by assumption, the curve α is smooth, thus its coordinates are smooth functions. Hence it is in principle possible to obtain a solution to this integral, although it may not always be a function that can be written down explicitly. Moreover, we see that, given a point $\tilde{\alpha}_z(t_0)$, the solution of (4.1) is unique. □

With these preliminaries, we are ready to prove the proposition. Note that, due to the choice of a different metric, our coefficients are slightly different from those of the reference.

Proposition 4.3. *[All98, Lemma 4.3] Any regular vertical path in $\mathbb{C}\mathcal{H}^{n+1}$ of length L is homotopic to a horizontal path of length $\sqrt{2\pi L}$ by a homotopy of area at most $2L + \frac{8\sqrt{\pi}}{3}L^{3/2}$.*

Proof. It is sufficient to provide the homotopy for $n = 1$, because for larger n, we obtain a horizontal path in $\mathbb{C}\mathcal{H}^{n+1}$ from the horizontal path in $\mathbb{C}\mathcal{H}^2$ by setting all other coordinates to zero.

We briefly explain the motivation for the proposed solution. In the context in which Allcock uses this result, it is desirable that the horizontal path to which the homotopy maps a given vertical path has minimum length. Considering that the z-coordinate of a horizontal path in coordinates is equal to twice the area enclosed by the projection of the path into $\mathbb{R}^2$, the goal of finding an appropriate horizontal path can therefore be reduced to lifting a path in $\mathbb{R}^2$ horizontally whose length in $\mathbb{R}^2$ is minimal while enclosing as much area as possible. The solution of this problem is easy: it is a circle.

We proceed to presenting the explicit construction of these paths together with the homotopy so that we can estimate its area. We start by defining the circles and then determine their horizontal lifts. Let S_t be the path in $\mathbb{R}^2$ that travels counterclockwise along the circle of radius t centred at $(-t, 0)$. This path traverses the origin at times $t = 0$ and $t = 2\pi$ and is given by

$$S_t(\theta) = \begin{pmatrix} -t \\ 0 \end{pmatrix} + t \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Note that the curve parameter is θ, while the radius t remains constant for a given S_t and will later take on the role of parameterising the vertical path.

From Lemma 4.2 we know that there are horizontal lifts $\tilde{S}_t$ of the paths S_t to $\mathbb{C}\mathcal{H}^2$. If we demand that the horizontal lifts $\tilde{S}_t$ pass through the origin at $t = 0$, they are unique and their explicit form can be obtained from (4.2). By carrying out the computation, we see that they are given by

$$\varphi(\tilde{S}_t(\theta)) = \begin{pmatrix} t\cos\theta - t \\ t\sin\theta \\ t^2(\theta - \sin\theta) \end{pmatrix}.$$

Any vertical path $\gamma\colon I \to \mathbb{C}\mathcal{H}^2$ satisfies

$$\gamma'(t) \in V_{\gamma(t)},$$

which implies that in coordinates, it has the form $\varphi(\gamma(t)) = (v_0, \gamma_z(t))$, where $v_0 \in \mathbb{R}^2$ is constant. We may assume $v_0 = 0$, otherwise we shift our coordinates accordingly. Thus every regular vertical path $\gamma\colon I \to \mathbb{C}\mathcal{H}^2$ starting at the origin is, up to reparameterisation, given by

$$\varphi(\gamma(t)) = \begin{pmatrix} 0 \\ 0 \\ 2\pi t^2 \end{pmatrix}.$$

It is important to note that we constructed the circles in $\mathbb{R}^2$ such that $\gamma(t)$ is precisely $\tilde{S}_t(2\pi)$. If we choose the domain of γ to be $I = [0, \sqrt{L/(2\pi)}]$, the vertical path has length L. We set $T = \sqrt{L/(2\pi)}$ and define the homotopy

$$\begin{aligned} \Gamma\colon [0,T] \times [0, 2\pi] &\to \mathbb{R} \\ (t, \theta) &\mapsto \tilde{S}_t(\theta). \end{aligned}$$

Given that $\Gamma(t, 2\pi) = \gamma(t)$ and $\Gamma(T, \theta) = \tilde{S}_T(\theta)$, it follows that the vertical path γ is homotopic to the horizontal path $\tilde{S}_T$.

To determine the length of $\tilde{S}_T$, we observe that on the horizontal subspaces in coordinates, the metric is simply the Euclidean metric. Hence the length of horizontal paths coincides with the

Euclidean length of their projections into $\mathbb{R}^2$. It follows that

$$\mathrm{Length}(\tilde{S}_T) = 2\pi T = \sqrt{2\pi L}.$$

Bounding the area of Γ is therefore all that remains to do to finish the proof. We first determine

$$\left\| \frac{\partial \Gamma}{\partial \theta} \right\| = \left\| \frac{\partial \tilde{S}_t}{\partial \theta} \right\| = \left\| \frac{\partial S_t}{\partial \theta} \right\| = t,$$

where the second equality follows because $\tilde{S}_t$ and thus $\frac{\partial \tilde{S}_t}{\partial \theta}$ is horizontal, and the third equality is a simple calculation in $\mathbb{R}^2$.

The other partial derivative has a horizontal and a vertical part, so that we write

$$\frac{\partial \Gamma}{\partial t} = h(t,\theta) + v(t,\theta),$$

where h is horizontal and v vertical. Using that the norm of the horizontal part coincides with the Euclidean norm of the projection of the horizontal part into $\mathbb{R}^2$, we see that

$$\|h(t,\theta)\| = \left\| \frac{\partial \tilde{S}_t}{\partial t} \right\| = \left\| \frac{\partial S_t}{\partial t} \right\| \leq 2.$$

The inequality follows from an explicit calculation of $\left\| \frac{\partial S_t}{\partial t} \right\|$ in $\mathbb{R}^2$.

For bounding the vertical part, recall that the vertical direction has the interpretation of measuring twice the area that is bounded by the projection of the path in coordinates onto $\mathbb{R}^2$ and radial line segments from its start- and endpoint to the origin. The projection of $\tilde{S}_t$ in coordinates onto its first two coordinates is obviously S_t. With this picture in mind, it is easy to see that

$$\|v(t,\theta)\| = 2\frac{\partial}{\partial t}\mathrm{Area}(D_{S_t|_{[0,\theta]}}) \leq 2\frac{\partial}{\partial t}\mathrm{Area}(D_{S_t|_{[0,2\pi]}}) = 4\pi t.$$

We omitted the absolute value because in the way that we arranged these curves, the area increases as t increases. It follows that

$$\left\| \frac{\partial \Gamma}{\partial t} \right\| \leq \|h(t,\theta)\| + \|v(t,\theta)\| \leq 4\pi t + 2.$$

We therefore obtain the following estimate for the area of the homotopy, writing $X = [0,T] \times [0,2\pi]$,

$$\begin{aligned}
\mathrm{Area}(\Gamma(X)) &\leq \int_X dt d\theta \left\| \frac{\partial \Gamma}{\partial t} \right\| \left\| \frac{\partial \Gamma}{\partial \theta} \right\| \\
&\leq \int_0^T dt (4\pi t + 2) t \int_0^{2\pi} d\theta \\
&= \frac{8\pi^2}{3} T^3 + 2\pi T^2 \\
&= \frac{8\sqrt{\pi}}{3} L^{3/2} + 2L.
\end{aligned}$$

□

4.2 The case of $\mathbb{H}\mathcal{H}^{n+1}$

In the final section of this thesis, we prove a generalisation of Proposition 4.3 to the quaternionic case for paths that are line segments. We show that this problem can be solved using a modification of the prior results.

The group under consideration in this section is $\mathbb{H}\mathcal{H}^{n+1}$. As for the complex case, we establish the setup first by choosing coordinates and expressing the horizontal and vertical subspaces in terms of these coordinates explicitly. While much remains very similar, there are now three vertical directions.

For choosing coordinates on $\mathbb{H}\mathcal{H}^{n+1}$ we proceed as in the previous section. Recall from Proposition 3.10 that the exponential map is a diffeomorphism. We can therefore choose a basis of the Lie algebra and define the coordinates on $\mathbb{H}\mathcal{H}^{n+1}$ through this basis, where the exponential map provides the necessary unique correspondence of elements of $\mathbb{H}\mathcal{H}^{n+1}$ to the Lie algebra elements. For $b \in \mathbb{H}^n$ and $s \in \operatorname{Im} \mathbb{H}$, let P_b and M_s be defined as above. There are $4n+3$ real basis vectors in the Lie algebra $\mathfrak{n}$. We choose the following basis vectors,

$$\begin{aligned}
U_l &= P_{e_l},\\
V_l &= P_{ie_l},\\
X_l &= P_{je_l},\\
Y_l &= P_{ke_l},\\
Z_i &= M_i,\\
Z_j &= M_j,\\
Z_k &= M_k,
\end{aligned}$$

where e_l denotes the l^{th} standard basis vector in $\mathbb{R}^n$. The nonvanishing brackets are

$$\begin{aligned}
[U_l, V_l] &= -2Z_i, && \text{for all } l = 1, \dots, n,\\
[X_l, Y_l] &= -2Z_i, && \text{for all } l = 1, \dots, n,\\
[U_l, X_l] &= -2Z_j, && \text{for all } l = 1, \dots, n,\\
[Y_l, V_l] &= -2Z_j, && \text{for all } l = 1, \dots, n,\\
[U_l, Y_l] &= -2Z_k, && \text{for all } l = 1, \dots, n,\\
[V_l, X_l] &= -2Z_k, && \text{for all } l = 1, \dots, n.
\end{aligned}$$

Since we have a basis of the Lie algebra of $\mathbb{H}\mathcal{H}^{n+1}$ that identifies its elements with elements of $\mathbb{H}^n \times \operatorname{Im} \mathbb{H}$, which again is identified with $\mathbb{R}^{4n+3}$, we can choose the following coordinates on $\mathbb{H}\mathcal{H}^{n+1}$,

$$\begin{aligned}
&\varphi\colon \mathbb{H}\mathcal{H}^{n+1} \to \mathbb{R}^{4n+3}\\
&\quad \exp\big(P_{(u_l+iv_l+jx_l+ky_l)_{l=1}^n} + M_{(iz_i+jz_j+kz_k)}\big) \mapsto (u_1, v_1, x_1, y_1, \cdots, u_n, v_n, x_n, y_n, z_i, z_j, z_k),
\end{aligned}$$

with parameterisation

$$\begin{aligned}\psi\colon \mathbb{R}^{4n+3} &\to \mathbb{H}\mathcal{H}^{n+1}\\ (u_1, v_1, x_1, y_1, \cdots, u_n, v_n, x_n, y_n, z_i, z_j, z_k) &\mapsto \exp(P_{(u_l+iv_l+jx_l+ky_l)_{l=1}^n} + M_{(iz_i+jz_j+kz_k)}).\end{aligned}$$

Due to the definition of the coordinates through the exponential map, we call these the *exponential coordinates* on $\mathbb{H}\mathcal{H}^{n+1}$.

A metric can again be declared by using the trace form as an inner product on $T_e\mathbb{H}\mathcal{H}^{n+1}$ and translating it to other points by left-multiplication. The basis vectors are orthogonal with respect to this inner product, and rescaling makes them orthonormal. We take the rescaled inner product to be g_e, and this can be translated to other points p by left multiplication L_p, so that $g_p = L_p^* g_e$.

Again, the choice of metric is not canonical and there are many different left-invariant metrics that we could have chosen instead, but for the previously outlined reasons, statements about lengths and areas given by a different metric would differ by at most some uniformly bounded amount, which could result in different prefactors in the statement of Proposition 4.4, but the same power laws in L.

We identify the horizontal and vertical subspaces next. Following Theorem 3.16, the horizontal subspace $H_e \subset T_e\mathbb{H}\mathcal{H}^{n+1}$ is given by $\mathfrak{g}_\alpha$, which is

$$\operatorname{span}_{\mathbb{R}}\{U_1, \cdots, U_n, V_1, \cdots, V_n, X_1, \cdots, X_n, Y_1, \cdots, Y_n\},$$

and this can be translated to other points $p \in \mathbb{H}\mathcal{H}^{n+1}$ with left-multiplication,

$$H_p = d_eL_pH_e = \operatorname{span}_{\mathbb{R}}\{d_eL_pU_1, \cdots, d_eL_pV_1, \cdots, d_eL_pX_1, \cdots, d_eL_pY_1, \cdots, d_eL_pY_n\}.$$

The vertical subspace at the point $p \in \mathbb{H}\mathcal{H}^{n+1}$ is the orthogonal complement $V_p = H_p^\perp$, that is,

$$V_p = \operatorname{span}_{\mathbb{R}}\{d_eL_pZ_i, d_eL_pZ_j, d_eL_pZ_k\} \quad \text{for all } p \in \mathbb{H}\mathcal{H}^{n+1}.$$

Note that the vertical subspaces are three-dimensional now.

As in the complex case, it is sufficient to focus on $\mathbb{H}\mathcal{H}^2$ from now on. In terms of our exponential coordinates, left multiplication becomes

$$(\varphi \circ L_{\psi(u,v,x,y,z_i,z_j,z_k)} \circ \psi)(a,b,c,d,e,f,g) = \begin{pmatrix} u+a \\ v+b \\ x+c \\ y+d \\ z_i + e + (ub - va) + (xd - yc) \\ z_j + f + (uc - xa) + (vd - by) \\ z_k + g + (ud - ya) + (vc - bx) \end{pmatrix},$$

with differential

$$d_{\varphi(e)}(\varphi \circ L_{\psi(u,v,x,y,z_i,z_j,z_k)} \circ \psi) = \begin{pmatrix} & I_4 & & & 0 \\ -v & u & -y & x & \\ -x & -y & u & v & I_3 \\ -y & -x & v & u & \end{pmatrix}.$$

The horizontal subspaces can be described as the intersection of the kernels of three 1-forms, one for each vertical direction. We denote these $\xi^{(i)}, \xi^{(j)}, \xi^{(k)}$, with the superscripts indicating the direction, and write them as a vector-valued differential form ξ, where

$$\xi_p = \begin{pmatrix} \xi_p^{(i)} \\ \xi_p^{(j)} \\ \xi_p^{(k)} \end{pmatrix}.$$

We set $\xi = \varphi^*\eta$, with

$$\eta_{(v,w,x,y,z_i,z_j,z_k)} = \begin{pmatrix} \eta^{(i)}_{(v,w,x,y,z_i,z_j,z_k)} \\ \eta^{(j)}_{(v,w,x,y,z_i,z_j,z_k)} \\ \eta^{(k)}_{(v,w,x,y,z_i,z_j,z_k)} \end{pmatrix} = \begin{pmatrix} dz_i - udv + vdu - xdy + ydx \\ dz_j - udx + xdu - vdy + ydv \\ dz_k - udy + ydu - vdx + xdv \end{pmatrix},$$

so that the horizontal subspaces are

$$H_p = \ker \xi_p.$$

A curve $\beta \colon I \to \mathbb{C}\mathcal{H}^2$ is horizontal if and only if $\beta'(t) \in \ker \xi_{\beta(t)}$ for all $t \in I$.

Like in the case of $\mathbb{C}\mathcal{H}^2$, the representation of the horizontal subspaces as kernels of the 1-form ξ yields a geometric interpretation for the z-coordinates of a horizontal curve in $\mathbb{H}\mathcal{H}^2$. However, this is more complicated and its discussion is deferred until it is required in the proof of Proposition 4.4.

Horizontal lifts $\tilde{\alpha}$ of a curve $\alpha \colon I \to \mathbb{R}^4$ are, as in the complex case, defined by the condition that

$$\pi(\tilde{\alpha}) = \alpha,$$

where $\pi \colon \mathbb{H}\mathcal{H}^2 \to \mathbb{R}^4, \psi(u,v,x,y,z_i,z_j,z_k) \mapsto (u,v,x,y)$. Lemma 4.2 can be generalised to prove the existence of such horizontal lifts, with the only difference that we now need to specify three initial conditions, one for each vertical directions, to obtain uniqueness of the horizontal lifts. If we write the horizontal curve $\tilde{\alpha}$ with $\pi(\tilde{\alpha}) = \alpha$ as

$$\varphi(\tilde{\alpha}(t)) = \begin{pmatrix} \tilde{\alpha}_u(t) \\ \tilde{\alpha}_v(t) \\ \tilde{\alpha}_x(t) \\ \tilde{\alpha}_y(t) \\ \tilde{\alpha}_{z_j}(t) \\ \tilde{\alpha}_{z_j}(t) \\ \tilde{\alpha}_{z_k}(t) \end{pmatrix},$$

then the condition $\tilde{\alpha}'(t) \in \ker \xi_{\tilde{\alpha}(t)}$ is equivalent to the system of equations

$$\begin{aligned}
\tilde{\alpha}_{z_i}(t) &= \tilde{\alpha}_u(t)\tilde{\alpha}'_v(t) - \tilde{\alpha}_v(t)\tilde{\alpha}'_u(t) + \tilde{\alpha}_x(t)\tilde{\alpha}'_y(t) - \tilde{\alpha}_y(t)\tilde{\alpha}'_x(t), \\
\tilde{\alpha}_{z_j}(t) &= \tilde{\alpha}_u(t)\tilde{\alpha}'_x(t) - \tilde{\alpha}_x(t)\tilde{\alpha}'_u(t) + \tilde{\alpha}_v(t)\tilde{\alpha}'_y(t) - \tilde{\alpha}_y(t)\tilde{\alpha}'_v(t), \\
\tilde{\alpha}_{z_k}(t) &= \tilde{\alpha}_u(t)\tilde{\alpha}'_y(t) - \tilde{\alpha}_y(t)\tilde{\alpha}'_u(t) + \tilde{\alpha}_v(t)\tilde{\alpha}'_x(t) - \tilde{\alpha}_x(t)\tilde{\alpha}'_v(t).
\end{aligned}$$

Obviously, we have $(\tilde{\alpha}_u(t), \tilde{\alpha}_v(t), \tilde{\alpha}_x(t), \tilde{\alpha}_y(t)) = (\alpha_u(t), \alpha_v(t), \alpha_x(t), \alpha_y(t))$, and the other components $(\tilde{\alpha}_{z_i}, \tilde{\alpha}_{z_j}, \tilde{\alpha}_{z_k})$ are given by

$$\begin{pmatrix} \tilde{\alpha}_{z_i}(t) - \tilde{\alpha}_{z_i}(t_0) \\ \tilde{\alpha}_{z_j}(t) - \tilde{\alpha}_{z_j}(t_0) \\ \tilde{\alpha}_{z_k}(t) - \tilde{\alpha}_{z_k}(t_0) \end{pmatrix} = \begin{pmatrix} \int_{t_0}^t \alpha_u(s)\alpha'_v(s) - \alpha_v(s)\alpha'_u(s) + \alpha_x(s)\alpha'_y(s) - \alpha_y(s)\alpha'_x(s) ds \\ \int_{t_0}^t \alpha_u(s)\alpha'_x(s) - \alpha_x(s)\alpha'_u(s) + \alpha_v(s)\alpha'_y(s) - \alpha_y(s)\alpha'_v(s) ds \\ \int_{t_0}^t \alpha_u(s)\alpha'_y(s) - \alpha_y(s)\alpha'_u(s) + \alpha_v(s)\alpha'_x(s) - \alpha_x(s)\alpha'_v(s) ds \end{pmatrix}. \tag{4.3}$$

With these preliminary considerations, we can prove the following proposition that generalises Proposition 4.3 for special vertical paths.

Proposition 4.4. *Any regular vertical path following a line segment in $\mathbb{H}\mathcal{H}^{n+1}$ of length L is homotopic to a horizontal path of length $\sqrt{2\pi L}$ by a homotopy of area at most $2L + 8\sqrt{\frac{\pi}{3}}L^{3/2}$.*

Proof. As in the proof of Proposition 4.3, it is sufficient to find the homotopy for the case that $n = 1$ only, because this allows us to obtain a suitable homotopy (with the same area) by setting all other coordinates of the horizontal path to zero. Any vertical path $\gamma\colon I \to \mathbb{H}\mathcal{H}^2$ must satisfy

$$\gamma'(t) \in V_{\gamma(t)},$$

which implies that in coordinates, it has the form

$$\varphi(\gamma(t)) = \begin{pmatrix} v_0 \\ \gamma_z(t) \end{pmatrix}, \quad \text{with } \gamma_z(t) = \begin{pmatrix} \gamma_{z_i}(t) \\ \gamma_{z_j}(t) \\ \gamma_{z_k}(t) \end{pmatrix},$$

where $v_0 \in \mathbb{R}^4$ is constant. We may assume $v_0 = 0$, otherwise we shift our coordinate system accordingly. We require that the remaining vertical components $\gamma_z(t)$ describe a regular path along a line segment, which implies that, up to reparameterisation, we can write $\gamma_z(t)$ as

$$\gamma_z(t) = 2\pi t^2 \begin{pmatrix} r_i \\ r_j \\ r_k \end{pmatrix},$$

where $r := (r_i, r_j, r_k) \in \mathbb{R}^3$ is constant and can be chosen such that $\|r\| = 1$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^3$.

As in the proof of Proposition 4.3, our goal is to find a family of paths $\{S_t\}_t$ in $\mathbb{R}^4$ that can be lifted to a family of horizontal paths $\{\tilde{S}_t\}_t$ in $\mathbb{H}\mathcal{H}^2$ such that $\gamma(t) = \tilde{S}_t(2\pi)$ yields a vertical path following a line segment.

We denote the paths as $S_t(\theta) = \alpha_t(\theta)$, where $\alpha_t = (\alpha_u, \alpha_v, \alpha_x, \alpha_y)_t$ with the subscript indicating the correspondence to S_t. To enhance readability, we often place the subscript t outside the brackets; this should be understood as assigning a subscript t to each component. This notation is employed because the coordinate functions will generally vary for different paths S_t and their horizontal lifts $\tilde{S}_t$. The horizontal lift $\tilde{S}_t$ must satisfy

$$\varphi(\tilde{S}_t(\theta)) = \begin{pmatrix} \alpha(\theta) \\ \tilde{\alpha}_{z_i}(\theta) \\ \tilde{\alpha}_{z_j}(\theta) \\ \tilde{\alpha}_{z_k}(\theta) \end{pmatrix}_t,$$

where $(\tilde{\alpha}_{z_i}, \tilde{\alpha}_{z_j}, \tilde{\alpha}_{z_k})_t$ is determined by the condition that $\frac{\partial}{\partial\theta}(\tilde{S}_t(\theta)) \in \ker \xi_{\tilde{S}_t(\theta)}$. We may set $\alpha_{y,t} = 0$ for all paths α_t in $\mathbb{R}^4$. This simplifies the differential equation determining $(\tilde{\alpha}_{z_i}, \tilde{\alpha}_{z_j}, \tilde{\alpha}_{z_k})_t$ that needs to be solved, which becomes

$$\begin{pmatrix} \tilde{\alpha}'_{z_i}(\theta) \\ \tilde{\alpha}'_{z_j}(\theta) \\ \tilde{\alpha}'_{z_k}(\theta) \end{pmatrix}_t = \begin{pmatrix} \alpha'_v(\theta)\alpha_u(\theta) - \alpha'_u(\theta)\alpha_v(\theta) \\ \alpha'_x(\theta)\alpha_u(\theta) - \alpha'_u(\theta)\alpha_x(\theta) \\ \alpha'_x(\theta)\alpha_v(\theta) - \alpha'_v(\theta)\alpha_x(\theta) \end{pmatrix}_t.$$

We rewrite this equation using the cross product in $\mathbb{R}^3$,

$$\begin{pmatrix} \tilde{\alpha}'_{z_i}(\theta) \\ -\tilde{\alpha}'_{z_j}(\theta) \\ \tilde{\alpha}'_{z_k}(\theta) \end{pmatrix}_t = \begin{pmatrix} \alpha'_x(\theta) \\ \alpha'_v(\theta) \\ \alpha'_u(\theta) \end{pmatrix}_t \times \begin{pmatrix} \alpha_x(\theta) \\ \alpha_v(\theta) \\ \alpha_u(\theta) \end{pmatrix}_t. \tag{4.4}$$

Observe the negative sign in the second entry and the reversed order of the components of α in the vectors on the right side of the equation, which become evident upon expanding the cross product.

Similar to the complex case, we aim to find a path α_t yielding the solution $(\tilde{\alpha}_{z_i}, \tilde{\alpha}_{z_j}, \tilde{\alpha}_{z_k})_t$ of the differential equation such that the following conditions are fulfilled:

$$\begin{pmatrix} \tilde{\alpha}_{z_i}(2\pi) \\ \tilde{\alpha}_{z_j}(2\pi) \\ \tilde{\alpha}_{z_k}(2\pi) \end{pmatrix}_t = \gamma_z(t) = 2\pi t^2 \begin{pmatrix} r_i \\ r_j \\ r_k \end{pmatrix} \quad \text{and} \quad \alpha_t(2\pi) = 0,$$

because then, we have $\tilde{S}_t(2\pi) = \gamma(t)$.

When $r = (r_1, r_2, r_3) = (1, 0, 0)$, only one nontrivial differential equation remains, which, omitting the subscript t, takes the form $\tilde{\alpha}'_{z_i}(\theta) = \alpha'_v(\theta)\alpha_u(\theta) - \alpha'_u(\theta)\alpha_v(\theta)$. From the proof of Proposition 4.3, we know there exist functions $(\alpha_u, \alpha_v)_t$ such that, after specifying boundary conditions by requiring that the path $\tilde{S}_t$ traverses the origin at time $t = 0$, the left side of the differential equation integrates to $(\tilde{\alpha}_{z_i}, \tilde{\alpha}_{z_j}, \tilde{\alpha}_{z_k})_t$ with $\tilde{\alpha}_{z_i,t}(2\pi) = 2\pi t^2$ and $\tilde{\alpha}_{z_j,t}(2\pi) = \tilde{\alpha}_{z_k,t}(2\pi) = 0$, and we have $(\alpha_u(2\pi), \alpha_v(2\pi))_t = (0, 0)$. By setting $\alpha_{x,t}$ to zero, we therefore define a path S_t, given by

$$S_t(\theta) = \begin{pmatrix} -t \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ 0 \end{pmatrix},$$

such that its horizontal lift $\tilde{S}_t$, when requiring that it passes through the origin at $t = 0$, is

$$\varphi(\tilde{S}_t(\theta)) = \begin{pmatrix} t\cos\theta - t \\ t\sin\theta \\ 0 \\ 0 \\ t^2(\theta - \sin\theta) \\ 0 \\ 0 \end{pmatrix},$$

Hence $\tilde{S}_t(2\pi)$ is a vertical path in the direction of the line segment $(1, 0, 0)$.

Now we consider the general case of a regular vertical path $\gamma\colon I \to \mathbb{H}\mathcal{H}^2$ following a vertical line segment $r = (r_1, r_2, r_3)$ with $\|r\| = 1$ which is, up to reparameterisation, given by

$$\varphi(\gamma(t)) = 2\pi t^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}.$$

We choose a rotation $R \in SO(3)$ such that $R(1, 0, 0) = (r_1, -r_2, r_3)$. The key observation that lets us use the solution for the special case $r = (1, 0, 0)$ to solve the problem for general r is that the differential equation behaves under rotations R in $SO(3)$ as follows,

$$R\left(\begin{pmatrix}\alpha_x'(\theta)\\ \alpha_v'(\theta)\\ \alpha_u'(\theta)\end{pmatrix}\times\begin{pmatrix}\alpha_x(\theta)\\ \alpha_v(\theta)\\ \alpha_u(\theta)\end{pmatrix}\right)=R\begin{pmatrix}\alpha_x'(\theta)\\ \alpha_v'(\theta)\\ \alpha_u'(\theta)\end{pmatrix}\times R\begin{pmatrix}\alpha_x(\theta)\\ \alpha_v(\theta)\\ \alpha_u(\theta)\end{pmatrix},$$

because due to the fact that R does not depend on θ, it is $(R(\alpha_x,\alpha_v,\alpha_u)^\intercal)' = R(\alpha_x',\alpha_v',\alpha_u')^\intercal$. This observation implies that once we have a solution $(\alpha_x,\alpha_v,\alpha_u)^\intercal$ for one particular r of our problem, we can use a rotation R in $SO(3)$ to obtain a solution $R(\alpha_x,\alpha_v,\alpha_u)^\intercal$ for a different r, because

$$R\begin{pmatrix}\tilde\alpha_{z_i}'(\theta)\\ -\tilde\alpha_{z_j}'(\theta)\\ \tilde\alpha_{z_k}'(\theta)\end{pmatrix}=R\left(\begin{pmatrix}\alpha_x'(\theta)\\ \alpha_v'(\theta)\\ \alpha_u'(\theta)\end{pmatrix}\times\begin{pmatrix}\alpha_x(\theta)\\ \alpha_v(\theta)\\ \alpha_u(\theta)\end{pmatrix}\right)=R\begin{pmatrix}\alpha_x'(\theta)\\ \alpha_v'(\theta)\\ \alpha_u'(\theta)\end{pmatrix}\times R\begin{pmatrix}\alpha_x(\theta)\\ \alpha_v(\theta)\\ \alpha_u(\theta)\end{pmatrix}.$$

In general, the determinant of R appears as a prefactor on one side of the equation, but we may leave this out because we assumed $R\in SO(3)$.

The transformation of α_t is not exactly a rotation by R, because we inverted the order of the entries in the vectors on the right side of the equation. Instead, we describe the transformation of α_t by

$$R_4=\begin{pmatrix}\reflectbox{R} & 0\\ 0 & 1\end{pmatrix}\in O(4),$$

where $\reflectbox{$R$}$ is defined by $(\reflectbox{$R$}(a_1,a_2,a_3))_l=(R(a_3,a_2,a_1))_{4-l}$ for $l=1,2,3$. More simply put, $\reflectbox{$R$}$ is the matrix that we obtain from inverting the order of the columns of R. Then, we have

$$\begin{pmatrix}(R_4\alpha)_x(\theta)\\ (R_4\alpha)_v(\theta)\\ (R_4\alpha)_u(\theta)\end{pmatrix}=R\begin{pmatrix}\alpha_x(\theta)\\ \alpha_v(\theta)\\ \alpha_u(\theta)\end{pmatrix},$$

and an analogous expression for α_t', which is precisely how the vectors on the right side of the differential equation (4.4) transform. Note that exchanging the first and third rows of R in the definition of $\reflectbox{$R$}$ leads to the determinant of $\reflectbox{$R$}$ and hence R_4 being negative.

That implies that, if $(\alpha_u,\alpha_v,\alpha_x)_t$ and the corresponding solution of (4.4), which is $(\tilde\alpha_{z_i},\tilde\alpha_{z_j},\tilde\alpha_{z_k})_t$, satisfy the conditions above for $r=(1,0,0)$, that is,

$$\begin{pmatrix}\tilde\alpha_{z_i}(2\pi)\\ \tilde\alpha_{z_j}(2\pi)\\ \tilde\alpha_{z_k}(2\pi)\end{pmatrix}_t=2\pi t^2\begin{pmatrix}1\\0\\0\end{pmatrix}\quad\text{and}\quad\alpha_t(2\pi)=0,$$

then the transformed path $(R_4\alpha)_t$ and the corresponding solution of the differential equation (4.4) which we denote by $(\widetilde{R_4\alpha_{z_i}}, \widetilde{R_4\alpha_{z_j}}, \widetilde{R_4\alpha_{z_k}})_t$ satisfy, after choosing suitable boundary conditions,

$$\begin{pmatrix} \widetilde{R_4\alpha_{z_i}}(2\pi) \\ \widetilde{R_4\alpha_{z_j}}(2\pi) \\ \widetilde{R_4\alpha_{z_k}}(2\pi) \end{pmatrix}_t = R \begin{pmatrix} \tilde{\alpha}_{z_i}(2\pi) \\ \tilde{\alpha}_{z_j}(2\pi) \\ \tilde{\alpha}_{z_k}(2\pi) \end{pmatrix}_t = R \begin{pmatrix} 2\pi t^2 \\ 0 \\ 0 \end{pmatrix} = 2\pi t^2 \begin{pmatrix} r_i \\ r_j \\ r_k \end{pmatrix} \quad \text{and} \quad (R_4\alpha)_t(2\pi) = 0.$$

The last equation follows from the linearity of R_4. The path $R_4 S_t$ thus lifts to the following horizontal path in $\mathbb{H}\mathcal{H}^2$ when requiring that it passes through at the origin at time $t = 0$,

$$\varphi(\widetilde{R_4 S_t}(\theta)) = \begin{pmatrix} \mathfrak{R} \begin{pmatrix} t\cos\theta - t \\ t\sin\theta \\ 0 \end{pmatrix} \\ 0 \\ R \begin{pmatrix} t^2(\theta - \sin\theta) \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \mathfrak{R} \begin{pmatrix} t\cos\theta - t \\ t\sin\theta \\ 0 \end{pmatrix} \\ 0 \\ t^2(\theta - \sin\theta) \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \end{pmatrix}.$$

It follows that $\widetilde{R_4 S_t}(2\pi) = \gamma(t)$. If we choose the domain to be $I = [0, \sqrt{L/(2\pi)}]$, the vertical path $\gamma(t)$ has length L. Let therefore $T = \sqrt{L/(2\pi)}$ and define the homotopy

$$\begin{aligned} \Gamma\colon [0,T] \times [0,2\pi] &\to \mathbb{R} \\ (t,\theta) &\mapsto \widetilde{R_4 S_t}(\theta). \end{aligned}$$

Given that $\Gamma(t, 2\pi) = \gamma(t)$ and $\Gamma(T, \theta) = \widetilde{R_4 S_T}(\theta)$, it follows that the vertical path γ is homotopic to the horizontal path $\widetilde{R_4 S_T}$.

To determine the length of $\tilde{S}_T$, we observe that on the horizontal subspaces in coordinates, the metric is simply the Euclidean metric. Hence the length of horizontal paths coincides with the Euclidean length of their projections into $\mathbb{R}^4$. Since R_4 leaves lengths in $\mathbb{R}^4$ invariant, it follows that

$$\text{Length}(\widetilde{R_4 S_T}) = \text{Length}(\tilde{S}_T) = 2\pi T = \sqrt{2\pi L}.$$

Bounding the area of the homotopy Γ is therefore all that remains to do to finish the proof. First we determine

$$\left\| \frac{\partial \Gamma}{\partial \theta} \right\| = \left\| \frac{\partial \widetilde{R_4 S_t}}{\partial \theta} \right\| = \left\| \frac{\partial R_4 S_t}{\partial \theta} \right\| = \left\| \frac{\partial S_t}{\partial \theta} \right\| = t.$$

The second equality follows because $\tilde{S}_t$ and therefore $\frac{\partial \tilde{S}_t}{\partial \theta}$ is horizontal, and the third equality follows because $R_4 \in O(4)$ is independent of θ and preserves the Euclidean metric.

The other partial derivative has a horizontal and a vertical part, so that we write

$$\frac{\partial \Gamma}{\partial t} = h(t,\theta) + v(t,\theta),$$

where h is horizontal and v vertical. Using that the norm of the horizontal part coincides with the norm induced by the Euclidean metric of the projection of the horizontal part into $\mathbb{R}^4$, we see that

$$\|h(t,\theta)\| = \left\|\frac{\partial \widetilde{R_4 S_t}}{\partial t}\right\| = \left\|\frac{\partial R_4 S_t}{\partial t}\right\| = \left\|\frac{\partial S_t}{\partial t}\right\| \leq 2.$$

The inequality follows from the analogous result for the complex case.

For bounding the vertical part, we can, in an analogy to the complex case, give the z_i, z_j, z_k-coordinates of a horizontal path the interpretations of measuring certain areas that are bounded by the projection of the path into planes inside the (u,v,x,y)-subspace (after potentially adding radial line segments to make it a closed path in each of these planes separately). More precisely, we could say that:

- The z_i-coordinate measures twice the sum of the areas in the (u,v)-plane and in the (x,y)-plane,
- The z_j-coordinate measures twice the sum of the areas in the (u,x)-plane and in the (v,y)-plane,
- The z_k-coordinate measures twice the sum of the areas in the (u,y)-plane and in the (v,x)-plane.

Our paths $\widetilde{R_4 S_t}$ are constant in the y-direction, hence there is no contribution from planes that contain the y-direction.

Since the projection of these paths in the (u,v,x)-subspace is a rotation of the circle that S_t describes in the (u,v)-plane, the areas in each of the $(u,v),(u,x),(v,x)$-planes is bounded by the area of that circle, which is given by πt^2.

With this picture in mind, it is easy to see that

$$\|v(t,\theta)\| \leq 2\sqrt{3}\left|\frac{\partial}{\partial t}\mathrm{Area}(D_{S_t|_{[0,\theta]}})\right| \leq 2\sqrt{3}\left|\frac{\partial}{\partial t}\mathrm{Area}(D_{S_t|_{[0,2\pi]}})\right| = 4\sqrt{3}\pi t.$$

It follows that

$$\left\|\frac{\partial \Gamma}{\partial t}\right\| \leq \|h(t,\theta)\| + \|v(t,\theta)\| \leq 4\sqrt{3}\pi t + 2.$$

We therefore find the following estimate for the area of the homotopy, writing $X = [0,T] \times [0,2\pi]$,

$$\begin{aligned}\mathrm{Area}(\Gamma(X)) &\leq \int_X dt d\theta \left\|\frac{\partial \Gamma}{\partial t}\right\| \left\|\frac{\partial \Gamma}{\partial \theta}\right\| \\ &\leq \int_0^T dt (4\sqrt{3}\pi t + 2)t \int_0^{2\pi} d\theta \\ &= \frac{8\pi^2}{\sqrt{3}}T^3 + 2\pi T^2 \\ &= 8\sqrt{\frac{\pi}{3}}L^{3/2} + 2L.\end{aligned}$$

□

Appendix

A A model for the tangent spaces

We show a construction that models the hyperbolic spaces $\mathbb{K}\mathbf{H}^n$ as subspaces of $\mathbb{K}^{n+1}$, where we identify $\mathbb{K}^{n+1}$ as a Riemannian manifold with $\mathbb{R}^{\dim_{\mathbb{R}}\mathbb{K}(n+1)}$ in the obvious way. Our construction shows that $T_{[x]}\mathbb{K}\mathbf{H}^n$ can be modelled by $x^\perp$ for any representative x of $[x]$, and that, in this model, the metric on $T_{[x]}\mathbb{K}\mathbf{H}^n$ may be taken to be the real part of the restriction of $\langle\cdot\,|\,\cdot\rangle$ to $x^\perp$ divided by $|\langle x\,|\,x\rangle|$. Moreover, we show a way to transform tangent vectors expressed in one model into another model and derive the Levi-Civita connection. We present the part of the construction involving computations in coordinates only for the quaternions, the other cases follow by setting for all quaternions $a+ib+jc+kd$ the parts c, d (if $\mathbb{K}=\mathbb{C}$) respectively b, c, d (if $\mathbb{K}=\mathbb{R}$) to zero.

We start by constructing a regular submanifold of $\mathbb{R}^{\dim_{\mathbb{R}}\mathbb{K}(n+1)}$ of which we then take a quotient and show that we can map it diffeomorphically onto $\mathbb{K}\mathbf{H}^n$. We only show this for the quaternions explicitly. To this end, we choose coordinates

$$\begin{aligned}\psi\colon \mathbb{H}^{n+1} &\to \mathbb{R}^{4(n+1)}\\ (a_1+ib_1+jc_1+kd_1,\cdots,a_{n+1}+ib_{n+1}+jc_{n+1}+kd_{n+1})&\\ \mapsto (a_1,b_1,c_1,d_1,\cdots,a_{n+1},b_{n+1},c_{n+1},d_{n+1}).&\end{aligned}$$

and define the maps

$$F\colon \mathbb{K}^{n+1}\to\mathbb{R},\quad x\mapsto\langle x\,|\,x\rangle,$$

and

$$\tilde F\colon \mathbb{R}^{4(n+1)}\to\mathbb{R},\quad x\mapsto (F\circ\psi^{-1})(x)=\langle\psi^{-1}(x)\,|\,\psi^{-1}(x)\rangle,$$

where $\langle\cdot\,|\,\cdot\rangle$ is given by Definition 2.2, that is, for $(a_1,b_1,c_1,d_1,\cdots,a_{n+1},b_{n+1},c_{n+1},d_{n+1})$ (where (a_1,b_1,c_1,d_1) corresponds to the quaternion $a_1+ib_1+jc_1+kd_1$), we evaluate $\tilde F$ as

$$\begin{aligned}\tilde F(a_1,b_1,c_1,d_1,\cdots,&a_{n+1},b_{n+1},c_{n+1},d_{n+1})\\ &=\sum_{i=2}^{n}\left(a_i^2+b_i^2+c_i^2+d_i^2\right)-2\left(a_1a_{n+1}+b_1b_{n+1}+c_1c_{n+1}+d_1d_{n+1}\right).\end{aligned}$$

In the following, we use F to emphasise relations in $\mathbb{H}^{n+1}$ and $\tilde F$ for computations in real coordinates.

Lemma 2.3 guarantees that F and $\tilde F$ are well-defined in the sense that their image is contained in the real numbers, this is also clear from the above derived expression. It is easy to read off the differential of $\tilde F$,

$$d_{(a_1,\cdots,d_{n+1})}\tilde F=(-2a_{n+1},-2b_{n+1},-2c_{n+1},-2d_{n+1},2a_2,\cdots,2d_n,-2a_1,-2b_1,-2c_1,-2d_1).$$

For all $(a_1,\cdots,d_{n+1}) \in \tilde{F}^{-1}(-1)$, the differential of $\tilde{F}$ is surjective because $\tilde{F}(0)=0$, hence 0 is not contained in $\tilde{F}^{-1}(-1)$. It follows that $\tilde{F}^{-1}(-1)$ is a regular submanifold of $\mathbb{R}^{4(n+1)}$.

We claim that at each point $(a_1,\cdots,d_{n+1}) \in \tilde{F}^{-1}(-1)$, it holds that

$$\ker d_{(a_1,\cdots,d_{n+1})}\tilde{F} = (a_1,\cdots,d_{n+1})^{\perp}, \tag{A.1}$$

where $y \in x^{\perp}$ whenever $\langle \psi^{-1}(x) \,|\, \psi^{-1}(y)\rangle = 0$. A simple calculation shows that, in fact, we have $(a_1',\cdots,d_{n+1}') \in (a_1,\cdots,d_{n+1})^{\perp}$, if and only if

$$\sum_{i=2}^{n}(a_i - ib_i - jc_i - kd_i)(a_i' + ib_i' + jc_i' + kd_i') = 2\left(a_1a_{n+1}' + b_1b_{n+1}' + c_1c_{n+1}' + d_1d_{n+1}'\right).$$

Note that the right-hand side of the equation is real, therefore it is without further calculations obvious that the imaginary part of the left-hand side must also vanish. We simplify the equation to see that $(a_1', d_{n+1}') \in (a_1,\cdots,d_{n+1})^{\perp}$ if and only if

$$\sum_{i=2}^{n}(a_ia_i' + b_ib_i' + c_ic_i' + d_id_i') - 2\left(a_1a_{n+1}' + b_1b_{n+1}' + c_1c_{n+1}' + d_1d_{n+1}'\right) = 0.$$

It follows that the left-hand side of the equation is

$$d_{(a_1,\cdots,d_{n+1})}\tilde{F}(a_1',\cdots,d_{n+1}')^{\mathsf{T}} = \sum_{i=2}^{n}(a_ia_i' + b_ib_i' + c_ic_i' + d_id_i') - 2\left(a_1a_{n+1}' + b_1b_{n+1}' + c_1c_{n+1}' + d_1d_{n+1}'\right),$$

which proves (A.1).

From now on, we assume $\mathbb{K} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$ again, where we treat F and $\tilde{F}$ as placeholders for the maps defined above as well as their analogues for the real and complex case.

Obviously, the canonical projection $\pi\colon \mathbb{K}^{n+1} \to \mathbb{K}\mathbf{P}^n$ maps the set $F^{-1}(-1)$ to $\mathbb{K}\mathbf{H}^n$ because if $x \in F^{-1}(-1)$, then $\langle x \,|\, x\rangle = -1$ is negative. The restriction $\pi|_{F^{-1}(-1)}$ as a map onto $\mathbb{K}\mathbf{H}^n$ is surjective, because given some $[x] \in \mathbb{K}\mathbf{H}^n$, we can set $\tilde{x} = \frac{x}{\sqrt{|\langle x|x\rangle|}}$, which is a preimage of $[x]$ under π because we have $\langle \tilde{x} \,|\, \tilde{x}\rangle = -1$ and $[\tilde{x}] = [x]$. However, it is easy to see that the projection is not injective, because it holds that $[x] = [x\lambda]$ for all $\lambda \in \mathbb{K} \setminus \{0\}$, and if $|\lambda| = 1$, then x and $x\lambda$ are (distinct) preimages of $[x]$ under π in $F^{-1}(-1)$.

To obtain a diffeomorphism, we need to circumvent the non-uniqueness in choosing a preimage under π. We therefore introduce the set $\mathcal{S} = \{\lambda \in \mathbb{K}\colon\ |\lambda| = 1\}$ and pass to the quotient

$$M := {F^{-1}(-1)}\big/{\mathcal{S}}.$$

By $\pi_{\mathcal{S}}$ we denote the canonical projection

$$\pi_{\mathcal{S}}\colon F^{-1}(-1) \to {F^{-1}(-1)}\big/{\mathcal{S}}.$$

In the next step, we construct a right inverse for it. Let $x \in F^{-1}(-1)$. Define

$$\tau_x \colon M \to F^{-1}(-1), \quad [y] \mapsto y\langle x \,|\, y\rangle^{-1} \frac{|\langle x \,|\, y\rangle|}{\sqrt{|\langle y \,|\, y\rangle|}}.$$

Note that the meaning of our brackets has changed now because $[y]$ is an element of M and not $\mathbb{K}\mathbf{H}^n$. In the following, it will always be clear from the context which bracket is used.

We need to prove that the map τ_x is well-defined. From Lemma 2.5 we know that $|\langle x \,|\, y\rangle|^2 \geq \langle x \,|\, x\rangle\langle y \,|\, y\rangle = 1$. This implies $\langle x \,|\, y\rangle \neq 0$. Clearly, $\langle \tau_x([y]) \,|\, \tau_x([y])\rangle = -1$. Moreover, consider two representatives y and $\tilde{y} = y\alpha$ for some $\alpha \in \mathcal{S}$ with $|\alpha| = 1$. Then

$$\begin{aligned} \tau_x([y]) &= \tilde{y}\langle x \,|\, \tilde{y}\rangle^{-1} \frac{|\langle x \,|\, \tilde{y}\rangle|}{\sqrt{|\langle \tilde{y} \,|\, \tilde{y}\rangle|}} = y\alpha(\langle x \,|\, y\rangle\alpha)^{-1} \frac{|\langle x \,|\, y\rangle|\,|\alpha|}{\sqrt{|\langle y \,|\, y\rangle|\,|\alpha|^2}} \\ &= y\alpha\alpha^{-1}\langle x \,|\, y\rangle^{-1} \frac{|\langle x \,|\, y\rangle|}{\sqrt{|\langle y \,|\, y\rangle|}} = y\langle x \,|\, y\rangle^{-1} \frac{|\langle x \,|\, y\rangle|}{\sqrt{|\langle y \,|\, y\rangle|}}, \end{aligned}$$

so that the mapping rule is independent of the representative that was chosen. It follows that $d_{[y]}\tau_x \colon T_{[y]}M \to T_{\tau_x([y])}F^{-1}(-1)$ is a right inverse of $d_{\tau_x([y])}\pi_{\mathcal{S}}$.

We define a metric on M as follows. Let $y \in M$ and $x \in \pi_{\mathcal{S}}^{-1}(y)$. Let $v, w \in T_{[y]}M$, denote by $g^{eucl.}$ the Riemannian metric on $\tilde{F}^{-1}(-1)$ (which is just the restriction of the standard Euclidean metric on $\mathbb{R}^{\dim_{\mathbb{R}} \mathbb{K}(n+1)}$ to $\tilde{F}^{-1}(-1)$ because $\tilde{F}^{-1}(-1)$ is an embedded submanifold) and we define $\tilde{g} = \psi^* g^{eucl.}$. We set

$$g_y(v, w) = \tilde{g}_{\tau_x([y])}(d_{[y]}\tau_x v, d_{[y]}\tau_x w). \tag{A.2}$$

We need to show that this definition does not depend on the choice of x. To do so, let $x, x' \in \pi_{\mathcal{S}}^{-1}(y)$. Then there is some $\lambda \in \mathcal{S}$ such that $x' = x\lambda = R_\lambda(x)$, where R_λ denotes the right-multiplication.

It is easy to see that $\pi_{\mathcal{S}}(x) = \pi_{\mathcal{S}}(x') = \pi_{\mathcal{S}}(R_\lambda(x))$ and hence

$$d_x\pi_{\mathcal{S}} = d_{x\lambda}\pi_{\mathcal{S}} \cdot d_x R_\lambda.$$

so that, since $d_x R_\lambda$ is invertible, if $d_{[y]}\tau_x$ is a right inverse of $d_{\tau_x([y])}\pi_{\mathcal{S}}$, then $d_{\tau_x([y])}R_\lambda d_{[y]}\tau_x$ is a right inverse of $d_{x\lambda}\pi_{\mathcal{S}}$.

Considering that for $\lambda \in \mathcal{S}$ it holds that $\bar{\lambda}\lambda = 1$, hence $\bar{\lambda} = \lambda^{-1}$, we see that

$$\begin{aligned} R_\lambda\tau_x([y]) &= \mathcal{N} y\langle x \,|\, y\rangle^{-1}\lambda \\ &= \mathcal{N} y(\lambda^{-1}\langle x \,|\, y\rangle)^{-1} \\ &= \mathcal{N} y(\langle x\lambda \,|\, y\rangle)^{-1} = \tau_{x\lambda}([y]). \end{aligned}$$

Note that the normalisation factor $\mathcal{N} = \frac{|\langle x|y\rangle|}{\sqrt{|\langle y|y\rangle|}}$ is invariant under $x \to x\lambda$ when $|\lambda| = 1$.

This yields

$$\begin{aligned}\tilde{g}_{\tau_{x\lambda}([y])}(d_{[y]}\tau_{x\lambda}v, d_{[y]}\tau_{x\lambda}w) &= \tilde{g}_{R_\lambda\tau_x([y])}(d_xR_\lambda d_{[y]}\tau_xv, d_xR_\lambda d_{[y]}\tau_xw)\\ &= (R_{\lambda^{-1}})^*\tilde{g}_{\tau_x([y])}(d_{[y]}\tau_xv, d_{[y]}\tau_xw),\end{aligned}$$

and since $d_xR_\lambda\colon x^\perp \to (x\lambda)^\perp$ is an isometry, the independence of the chosen preimage follows. Nondegeneracy is clear because as a right inverse of $\pi_{\mathcal{S}}$, the map τ_x must be injective. The remaining properties are also inherited from the metric $\tilde{g}$.

We conclude that there is a diffeomorphism between the Riemannian manifolds

$$F^{-1}(-1)/\mathcal{S} \cong \mathbb{K}\mathbf{H}^n,$$

and that for any $[y]$ in $F^{-1}(-1)/\mathcal{S}$, a metric can be obtained using the standard metric inherited from $\mathbb{K}^{n+1}$ (i.e. the restriction of the quadratic form $\langle\cdot\,|\,\cdot\rangle$ to $x^\perp$ for any $x \in \pi_{\mathcal{S}}^{-1}$). Hence $x^\perp$ with metric $g_x = \langle\cdot\,|\,\cdot\rangle|_{x^\perp}$ is a model for the tangent space $T_{[x]}\mathbb{K}\mathbf{H}^n$.

The fact that models of $\mathbb{K}\mathbf{H}^n$ constructed with different immersions τ_x and $\tau_{x\lambda}$ are related by right multiplication R_λ explains why in our models of the tangent space in Definition 2.6, a tangent vector $U \in T_{[y]}\mathbb{K}\mathbf{H}^n$ that is represented by $u \in y^\perp$ must be represented by $u\lambda \in (y\lambda)^\perp$ when changing the model.

If we want to obtain a model using a representative y of $[y]$ with $\langle y\,|\,y\rangle$ not necessarily equal to -1, we may use right multiplication by $\sqrt{|\langle y\,|\,y\rangle|}$ to obtain from the above-constructed model a model where we identify $T_{[y]}\mathbb{K}\mathbf{H}^n$ with $y^\perp$ by choosing some $x \in \pi_{\mathcal{S}}^{-1}(y)$ and equipping $T_{[y]}\mathbb{K}\mathbf{H}^n$ with the inner product

$$g_{[y]} = \left(R_{\sqrt{|\langle y|y\rangle|}^{-1}}\right)^*\tilde{g}_{\tau_x([y])}.$$

The metric thus differs from the one determined above by a prefactor of $\frac{1}{|\langle y|y\rangle|}$. It follows that statements involving the metric become independent of the choice of representative y of $[y] \in \mathbb{K}\mathbf{H}^n$ when, simultaneously with exchanging y by $\frac{y}{\sqrt{|\langle y|y\rangle|}}$, we also replace the model $u \in y^\perp$ of the tangent vector $U \in T_{[y]}\mathbb{K}\mathbf{H}^n$ by $\frac{u}{\sqrt{|\langle y|y\rangle|}}$.

Proposition A.1. *For all $x \in \mathcal{S}$, the Levi-Civita connection ∇ on $\tau_x(\mathbb{K}\mathbf{H}^n)$ is given by the tangential projection of the standard connection D on $\mathbb{K}^{n+1}$ (which is easily obtained from the standard connection on $\mathbb{R}^{\dim_{\mathbb{R}}\mathbb{K}(n+1)}$). More precisely, if X and Y are smooth vector fields on $\tau_x(\mathbb{K}\mathbf{H}^n)$ and if $\tilde{X}$ and $\tilde{Y}$ are smooth extensions of X and Y on $\mathbb{K}^{n+1}$, and we denote $M = \tau_x(\mathbb{K}\mathbf{H}^n)$, then the Levi-Civita connection ∇ on M is defined by*

$$\nabla_XY = \mathrm{proj}_{TM}D_{\tilde{X}}\tilde{Y}|_M. \tag{A.3}$$

Proof. First we note that (A.3) is a well-defined connection (e.g. pp. 124 in [Lee18]). To prove that ∇ is in fact the Levi-Civita connection, we need to show that ∇ is symmetric and compatible with the metric on $\tau_x(\mathbb{K}\mathbf{H}^n)$. Proving these two properties is easy once we know that τ_x is an isometric immersion, i.e. $\tau_x^* \tilde{g} = g$. This, however, follows immediately from the definition of the metric in (A.2).

We confirm metric compatibility first. Let X and Y be smooth vector fields on M and $\tilde{X}$ and $\tilde{Y}$ be smooth extensions of X and Y on $\mathbb{K}^{n+1}$. Then on M it holds that

$$\begin{aligned}
\nabla_X g(Y,Z) &= D_{\tilde{X}} \tilde{g}(\tilde{Y},\tilde{Z}) \\
&= \tilde{g}(D_{\tilde{X}}\tilde{Y},\tilde{Z}) + \tilde{g}(\tilde{Y}, D_{\tilde{X}}\tilde{Z}) \\
&= \tilde{g}(\mathrm{proj}_{TM}\left(D_{\tilde{X}}\tilde{Y}\right),\tilde{Z}) + \tilde{g}(\tilde{Y}, \mathrm{proj}_{TM}\left(D_{\tilde{X}}\tilde{Z}\right)) \\
&= g(\nabla_X Y, Z) + g(Y, \nabla_X Z),
\end{aligned}$$

where the second equality follows from the metric compatibility of D and the third from the fact that $\tilde{Y}$ and $\tilde{Z}$ are tangent to M. Hence ∇ is compatible with the metric g.

To prove that ∇ is also symmetric, we first note that the naturality of Lie brackets implies that $[\tilde{X},\tilde{Y}]$ is tangent to M because the inclusion of $[X,Y]$ into $\mathbb{K}^{n+1}$ is simply given by the Lie bracket of the inclusions of X and Y which is clearly tangent to M. Therefore, on M, we see that

$$\begin{aligned}
\nabla_X Y - \nabla_Y X &= \mathrm{proj}_{TM}\left(D_{\tilde{X}}\tilde{Y}|_M - D_{\tilde{Y}}\tilde{X}|_M\right) \\
&= \mathrm{proj}_{TM}\left([\tilde{X},\tilde{Y}]|_M\right) \\
&= [\tilde{X},\tilde{Y}]|_M \\
&= [X,Y],
\end{aligned}$$

where the second equality follows from the symmetry of D and the third from the fact that $[\tilde{X},\tilde{Y}]$ is tangent to M. Hence ∇ is symmetric and therefore the Levi-Civita connection on $\tau_x(\mathbb{K}\mathbf{H}^n)$. $\square$

Remark. Note that for $\mathbb{K} = \mathbb{R}$, the set $\mathcal{S}$ contains only two elements, $\mathcal{S} = \{\pm 1\}$. The whole construction of an immersion τ_x of $\mathbb{R}\mathbf{H}^n$ into $\mathbb{R}^{n+1}$ can therefore be viewed as choosing one out of two copies of the hyperbolic space in $\mathbb{R}^{n+1}$. Given such a choice, we obtain immediately a metric on $\mathbb{R}\mathbf{H}^n$ and tangent spaces that are subsets of $\mathbb{R}^{n+1}$ without having to write down the immersion explicitly, and in fact, this is precisely the well-known hyperboloid model of $\mathbb{R}\mathbf{H}^n$.

B Construction of an orthonormal basis for quaternionic vector spaces

The proof of Proposition B.2 is a standard proof using the tools of linear algebra. However, due to the non-commutativity of the multiplication, the set of quaternions is not a field, so that $\mathbb{H}^{n+1}$ is not a vector space over a field and it is not obvious that the same tools can be applied. The set of quaternions is a division ring, which allows for the definition of (right) modules that have many of the properties of vector spaces over fields. We therefore call them (right) vector spaces and state the properties that are relevant for the following proofs. It is true that every such vector space has a basis of well-defined cardinality (its dimension) and every linearly independent set is contained in a basis. Furthermore, maximal linear independent set is a basis, and each spanning set contains a basis (pp. 180 in [Hun74]). The following proofs use these facts. Many results that are well-known for vector spaces over fields carry over to vector spaces over division rings, such as the following.

Lemma B.1. *[Hun74, Corollary 2.15 in Chapter IV] Let V, W be subspaces of a vector space over a division ring of finite dimension. Then*

$$\dim V + \dim W = \dim(V \cap W) + \dim(V + W).$$

Our goal is to prove the following proposition.

Proposition B.2. *Let $\langle \cdot \,|\, \cdot \rangle$ be the as in Definition 2.2, and let $v_1 \in \mathbb{K}^{n,1}$ such that $\langle v_1 \,|\, v_1 \rangle < 0$. Then the restriction of $\langle \cdot \,|\, \cdot \rangle$ to $v^\perp = \{w \in \mathbb{K}^{n+1} \colon \langle v \,|\, w \rangle = 0\}$ is positive definite.*

From the definition of $\langle \cdot \,|\, \cdot \rangle$ it is clear that in the standard basis $(e_1, \cdots, e_{n+1})$ of $\mathbb{K}^{n+1}$, the quadratic form Q is represented by the matrix

$$K = \begin{pmatrix} & & -1 \\ & I_{n-1} & \\ -1 & & \end{pmatrix},$$

where I_{n-1} denotes the $(n-1) \times (n-1)$-unit matrix. We define a new basis $(f_1, \cdots, f_{n+1})$ by setting

$$f_i = \begin{cases} \frac{1}{\sqrt{2}}(e_1 - e_{n+1}) & \text{for } i = 1, \\ e_i & \text{for } 2 \leq i \leq n, \\ \frac{1}{\sqrt{2}}(e_1 + e_{n+1}) & \text{for } i = n+1. \end{cases}$$

A simple calculation shows that in this basis, the quadratic form is represented by the matrix

$$K' = \begin{pmatrix} I_n & \\ & -1 \end{pmatrix}.$$

Thus, we know that there exists a basis in which the quadratic form has signature $(n, 1)$. We prove two lemmas from which Proposition B.2 directly follows.

Lemma B.3. *Let $v_1 \in \mathbb{K}^{n,1}$ such that $\langle v_1 \,|\, v_1 \rangle < 0$ be given. Then there exists an orthogonal basis (with respect to $\langle \cdot \,|\, \cdot \rangle$) containing v_1.*

Proof. Let $(v_1, \cdots, v_{n+1})$ be a basis of $\mathbb{K}^{n,1}$. Suppose that $\langle v_1 \,|\, v_1 \rangle < 0$. We want to transform this into an orthogonal basis containing v_1.

We modify the well-known Gram-Schmidt algorithm to find such a basis. The problem with the Gram-Schmidt method here is that, due to the possibility of nullvectors, we cannot always divide by the norm of a vector. This difficulty can be circumvented by making the vectors with nonzero norm pairwise orthogonal first using the Gram-Schmidt algorithm, and treating the nullvectors separately. In the first step, we prove by induction, that the basis $(v_1, \cdots, v_{n+1})$ can be transformed into a basis consisting of two sets N and O, where N contains all nullvectors and O the remaining basis vectors, and the vectors in O are pairwise orthogonal.

Instead of proving the lemma, we prove the slightly different claim that given some v_1 with $\langle v_1 \,|\, v_1 \rangle < 0$, there exists an orthogonal basis containing a scalar multiple of v_1. Then, the replacement of this scalar multiple by v_1 clearly yields an orthogonal basis as required in the lemma.

We prove the claim for the case $n = 1$ first. Set

$$b_1 = v_1 \frac{1}{\sqrt{|\langle v_1 \,|\, v_1 \rangle|}}$$

and

$$b_2 = v_2 + b_1 \langle b_1 \,|\, v_2 \rangle.$$

Then b_1 and b_2 are still linearly independent because v_1 and v_2 were so, and they are pairwise orthogonal because

$$\langle b_1 \,|\, b_2 \rangle = \langle b_1 \,|\, v_2 \rangle + \langle b_1 \,|\, b_1 \rangle \langle b_1 \,|\, v_2 \rangle = 0.$$

If $\langle b_2 \,|\, b_2 \rangle \neq 0$, then we set $O = \{b_1, b_2\}$, otherwise we set $O = \{b_1\}$ and $N = \{b_2\}$. In either case, the claim is proven for $n = 1$.

We return to considering arbitrary $n \in \mathbb{N}$. Let $i \leq n$ and suppose that $b_1, \cdots, b_i$ are already pairwise orthogonal or nullvectors. Let $O = \{b_k \colon \langle b_k \,|\, b_k \rangle \neq 0\} \subset \{1, \cdots, i\}$. Then we set

$$b_{i+1} = v_{i+1} - \sum_{b_k \in O} b_k \frac{\langle b_k \,|\, v_{i+1} \rangle}{\langle b_k \,|\, b_k \rangle}.$$

For any $1 < j \leq i$ with $b_j \in O$ it holds that

$$\langle b_j \,|\, b_{i+1} \rangle = \langle b_j \,|\, v_{i+1} \rangle - \sum_{b_k \in O} \langle b_j \,|\, b_k \rangle \frac{\langle b_k \,|\, v_{i+1} \rangle}{\langle b_k \,|\, b_k \rangle} = \langle b_j \,|\, v_{i+1} \rangle - \langle b_j \,|\, b_j \rangle \frac{\langle b_j \,|\, v_{i+1} \rangle}{\langle b_j \,|\, b_j \rangle} = 0.$$

The linear independence of the vectors $v_1, \cdots, v_{i+1}$ guarantees the linear independence of our transformed basis. Hence there is always a basis with the properties that it contains v_1 and that the other basis vectors are either pairwise orthogonal or nullvectors. Suppose we have such a basis and let O be the set of pairwise orthogonal non-nullvectors. In order to transform this into an orthogonal basis, we first make the nullvectors orthogonal to all vectors in O and in a second step we orthogonalise the set of nullvectors.

To make the nullvectors $N = \{v_l : \langle v_l \,|\, v_l \rangle = 0\}$ orthogonal to O, we set for $v_l \in N$

$$b_l = v_l - \sum_{b_k \in O} b_k \frac{\langle b_k \,|\, v_l \rangle}{\langle b_k \,|\, b_k \rangle}.$$

Then for all b_j with $b_j \in O$, it holds that

$$\langle b_j \,|\, b_l \rangle = \langle b_j \,|\, v_l \rangle - \sum_{b_k \in O} \langle b_j \,|\, b_k \rangle \frac{\langle b_k \,|\, v_l \rangle}{\langle b_k \,|\, b_k \rangle} = \langle b_j \,|\, v_l \rangle - \langle b_j \,|\, b_j \rangle \frac{\langle b_k \,|\, v_l \rangle}{\langle b_l \,|\, b_l \rangle} = 0.$$

If the resulting vector b_l is not a nullvector, we apply the procedure above to make it orthogonal to O. To achieve that the remaining nullvectors are pairwise orthogonal, we can repeatedly apply the following. Let b_l, b_m be nullvectors and suppose that $\langle b_l \,|\, b_m \rangle \neq 0$. For all $\lambda \in \mathbb{K}$ we have

$$\langle b_l \pm b_m \lambda \,|\, b_l \mp b_m \lambda \rangle = 2\mathrm{Im}\,(\langle b_1 \,|\, b_2 \rangle \lambda).$$

If we choose $\lambda = \langle b_2 \,|\, b_1 \rangle$, then $\langle b_1 \,|\, b_2 \rangle \lambda = |\langle b_1 \,|\, b_2 \rangle|^2$ is purely real so that the two vectors $b_l + b_m\lambda$ and $b_l - b_m\lambda$ are orthogonal. The resulting vectors are certainly still linearly independent, this yields the desired orthogonal basis. □

The previous lemma guarantees the existence of a basis B containing v_1 whose basis vectors are pairwise orthogonal. For the proof of Proposition B.2, it remains to show that the signature is independent of the chosen basis (a result that is well-known for real and complex vector spaces). Since we already know the signature of Q in one basis, the proposition then easily follows.

Lemma B.4. *Let Q be a quadratic form on a quaternionic vector space. Then there is a basis in which the quadratic form is represented by the matrix*

$$S = \begin{pmatrix} I_{r_+} & & \\ & -I_{r_-} & \\ & & 0 \end{pmatrix},$$

and the the numbers r_+ and r_- are independent of the particular choice of such a basis.

Proof. Let $B = \{b_1, \cdots, b_{n+1}\}$ be an orthogonal basis with respect to the quadratic form. Set

$$w_i = \begin{cases} b_i & \text{if } \langle b_i \,|\, b_i \rangle = 0, \\ \frac{b_i}{\sqrt{|\langle b_i | b_i \rangle|}} & \text{else.} \end{cases}$$

In this basis, the quadratic form is, after relabelling the basis vectors, represented by the matrix

$$S = \begin{pmatrix} I_{r_+} & & \\ & -I_{r_-} & \\ & & 0 \end{pmatrix}.$$

We want to show that the numbers r_+ and r_- are independent of the basis chosen. Since we already know a basis in which the quadratic form has signature $(n, 1)$, it will follow that there are no nullvectors remaining after the above construction.

Since the rank of any matrix is independent of the chosen basis (pp. 187 in [Hun74]), it is sufficient to prove the claim for r_+.

Let $V_+ = \operatorname{span}\{v_i \colon \langle v_i \,|\, v_i\rangle > 0\}$ and define V_- and V_0 analogously. We set $a = \max\{\dim W : W \subset V, \langle w \,|\, w\rangle > 0 \text{ for all } w \in W \setminus \{0\}\}$. It is clear that $a \geq r_+$ because V_+ is such a subspace W. Assume $a > r_+$ and let W be a subspace with the corresponding property and maximal dimension. Then Lemma B.1 yields

$$\dim W + \dim V_- + \dim V_0 > 0,$$

which implies that

$$W \cap (V_- + V_0) \neq \{0\}.$$

But then, there exists some $w \neq 0$ such that $\langle w \,|\, w\rangle > 0$ and $\langle w \,|\, w\rangle \leq 0$, this is a contradiction. We therefore have $a = r_+$ so that r_+, and r_- are independent of the basis chosen. □

Proof of Proposition B.2. We can use Lemma B.3 to obtain an orthogonal basis containing v_1. From Lemma B.4 we know that the signature of the quadratic form Q is independent of the basis. Our preliminary considerations show that Q has signature $(n, 1)$, and since $\langle v_1 \,|\, v_1\rangle < 0$, it follows that the restriction of $\langle \cdot \,|\, \cdot \rangle$ to $v_1^{\perp}$ is positive definite. □

C The kernel of $d_{\hat{o}}\pi$

We provide the proof of Lemma 2.33 for the quaternionic case only here, because the real case was treated above and the proof for the complex case is analogous to the proof below.

Lemma C.1. *Let* $\pi\colon \mathbb{H}^{n+1} \to \mathbb{H}\mathbf{H}^n$ *be the canonical projection and* $\hat{o} = (1, 0, \cdots, 0, 1)$. *The kernel of* $d_{\hat{o}}\pi$ *is the following set,*

$$\ker d_{\hat{o}}\pi = \{(\lambda, 0, \cdots, 0, \lambda) \colon \lambda \in \mathbb{H}\} = \operatorname{span}_{\mathbb{H}}\{\hat{o}\}.$$

Proof. We first note that for any $[(x_1, \cdots, x_{n+1})] \in \mathbb{H}\mathbf{H}^n$ it holds that

$$\langle x \mid x \rangle = \sum_{i=2}^{n} |x_1|^2 - 2\mathrm{Re}\ (\overline{x}_1 x_{n+1}) < 0$$

and therefore $x_{n+1} \neq 0$, which implies that its inverse $x_{n+1}^{-1} = \frac{\overline{x_{n+1}}}{|x_{n+1}|^2}$ exists. Recall that for the real case we chose coordinates

$$\begin{aligned} \varphi_{\mathbb{R}} \colon \mathbb{R}\mathbf{H}^n &\to \mathbb{R}^n \\ [(x_1, \cdots, x_{n+1})] &\mapsto \left(x_1 x_{n+1}^{-1}, \cdots, x_n x_{n+1}^{-1}\right). \end{aligned}$$

A quaternion can be described with four real coordinates,

$$(a, b, c, d) \mapsto a + ib + jc + kd.$$

This yields coordinates for $\mathbb{H}^{n+1}$ by defining

$$\begin{aligned} \psi \colon \mathbb{H}^{n+1} &\to \mathbb{R}^{4(n+1)} \\ (a_1 + ib_1 + jc_1 + kd_1, \cdots, a_{n+1} &+ ib_{n+1} + jc_{n+1} + kd_{n+1}) \\ &\mapsto (a_1, b_1, c_1, d_1, \cdots, a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}). \end{aligned}$$

We can modify our coordinates given by $\varphi_{\mathbb{R}}$ from the proof of Lemma 2.33 by applying the mapping rule of $\varphi_{\mathbb{R}}$ to quaternions, carefully separating the real and imaginary parts and using an analogue of ψ defined on $\mathbb{H}^n$ to map the result into $\mathbb{R}^{4n}$. This yields coordinates given by

$$\begin{aligned} \varphi \colon \mathbb{H}\mathbf{H}^n &\to \mathbb{R}^{4n} \\ &[(a_1 + ib_1 + jc_1 + kd_1, \cdots, a_{n+1} + ib_{n+1} + jc_{n+1} + kd_{n+1})] \\ &\mapsto \Bigg(\frac{a_1 a_{n+1} + b_1 b_{n+1} + c_1 c_{n+1} + d_1 d_{n+1}}{a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2}, \frac{-a_1 b_{n+1} + b_1 a_{n+1} - c_1 d_{n+1} + d_1 c_{n+1}}{a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2}, \\ &\quad \frac{-a_1 c_{n+1} + c_1 a_{n+1} - d_1 b_{n+1} + b_1 d_{n+1}}{a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2}, \frac{-a_1 d_{n+1} + d_1 a_{n+1} - c_1 b_{n+1} + b_1 c_{n+1}}{a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2}, \\ &\quad \cdots, \frac{-a_n d_{n+1} + d_n a_{n+1} - c_n b_{n+1} + b_n c_{n+1}}{a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2} \Bigg). \end{aligned}$$

The fact that the definition of φ is independent of the representative is less obvious than in the real case, but it can be seen without carrying out the computation in coordinates. Consider multiplying $(x_1, \cdots, x_{n+1}) \in \mathbb{H}^{n+1}$ by some $\lambda \in \mathbb{H} \setminus \{0\}$. Applying the mapping rule of $\varphi_{\mathbb{R}}$ to the result yields

$$(x_1\lambda)(x_{n+1}\lambda)^{-1} = x_1\lambda\frac{\overline{x_{n+1}\lambda}}{|x_{n+1}\lambda|^2} = x_1\lambda\overline{\lambda}\frac{\overline{x_{n+1}}}{|\lambda|^2\,|x_{n+1}|^2} = x_1\,|\lambda|^2\,\frac{\overline{x_{n+1}}}{|\lambda|^2\,|x_{n+1}|^2} = x_1\frac{\overline{x_{n+1}}}{|x_{n+1}|^2} = x_1x_{n+1}^{-1},$$

in the first component and analogous expressions in the other components, so that the definition of φ is independent of the chosen representative.

In terms of these coordinates, the canonical projection π can be expressed as

$$\begin{aligned}
&\varphi(\pi(\psi^{-1}(a_1, b_1, c_1, d_1, \cdots, a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}))) = \\
&\Bigg(\frac{a_1a_{n+1} + b_1b_{n+1} + c_1c_{n+1} + d_1d_{n+1}}{a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2}, \frac{-a_1b_{n+1} + a_{n+1}b_1 - c_1d_{n+1} + d_1c_{n+1}}{a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2}, \\
&\frac{-a_1c_{n+1} + c_1a_{n+1} - d_1b_{n+1} + b_1d_{n+1}}{a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2}, \frac{-a_1d_{n+1} + d_1a_{n+1} - c_1b_{n+1} + b_1c_{n+1}}{a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2}, \\
&\qquad\qquad \cdots, \frac{-a_nd_{n+1} + d_na_{n+1} - c_nb_{n+1} + b_nc_{n+1}}{a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2}\Bigg).
\end{aligned}$$

The differential of π at any point $p = (a_1, b_1, c_1, d_1, \cdots, d_{n+1})$ can now be computed explicitly. We write the differential as a block matrix,

$$d_p(\varphi \circ \pi \circ \psi^{-1}) = \begin{pmatrix} B & 0 & \cdots & 0 & E_1 \\ 0 & B & \cdots & 0 & E_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & B & E_n \end{pmatrix},$$

where, writing $|x_{n+1}|^2$ instead of $a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2$ for better readability, we have

$$B = \begin{pmatrix} \frac{a_{n+1}}{|x_{n+1}|^2} & \frac{b_{n+1}}{|x_{n+1}|^2} & \frac{c_{n+1}}{|x_{n+1}|^2} & \frac{d_{n+1}}{|x_{n+1}|^2} \\ \frac{-b_{n+1}}{|x_{n+1}|^2} & \frac{a_{n+1}}{|x_{n+1}|^2} & \frac{-d_{n+1}}{|x_{n+1}|^2} & \frac{c_{n+1}}{|x_{n+1}|^2} \\ \frac{-c_{n+1}}{|x_{n+1}|^2} & \frac{d_{n+1}}{|x_{n+1}|^2} & \frac{a_{n+1}}{|x_{n+1}|^2} & \frac{-b_{n+1}}{|x_{n+1}|^2} \\ \frac{-d_{n+1}}{|x_{n+1}|^2} & \frac{c_{n+1}}{|x_{n+1}|^2} & \frac{-b_{n+1}}{|x_{n+1}|^2} & \frac{a_{n+1}}{|x_{n+1}|^2} \end{pmatrix},$$

and, denoting for $i = 1\cdots, n$ the component of $\varphi(\pi(\psi^{-1}(p)))$ corresponding to the a_i-coordinate as $f(p)_{a_i}$ (and using an analogous notation for b_i, c_i, d_i), we set

$$E_i = \begin{pmatrix} \frac{a_i - 2a_{n+1}f(p)_{a_i}}{|x_{n+1}|^2} & \frac{b_i - 2b_{n+1}f(p)_{a_i}}{|x_{n+1}|^2} & \frac{c_i - 2c_{n+1}f(p)_{a_i}}{|x_{n+1}|^2} & \frac{d_i - 2d_{n+1}f(p)_{a_i}}{|x_{n+1}|^2} \\ \frac{b_i - 2a_{n+1}f(p)_{b_i}}{|x_{n+1}|^2} & \frac{-a_i - 2b_{n+1}f(p)_{b_i}}{|x_{n+1}|^2} & \frac{d_i - 2c_{n+1}f(p)_{b_i}}{|x_{n+1}|^2} & \frac{-c_i - 2d_{n+1}f(p)_{b_i}}{|x_{n+1}|^2} \\ \frac{c_i - 2a_{n+1}f(p)_{c_i}}{|x_{n+1}|^2} & \frac{-d_i - 2b_{n+1}f(p)_{c_i}}{|x_{n+1}|^2} & \frac{-a_i - 2c_{n+1}f(p)_{c_i}}{|x_{n+1}|^2} & \frac{b_i - 2d_{n+1}f(p)_{c_i}}{|x_{n+1}|^2} \\ \frac{d_i - 2a_{n+1}f(p)_{d_i}}{|x_{n+1}|^2} & \frac{-c_i - 2b_{n+1}f(p)_{d_i}}{|x_{n+1}|^2} & \frac{b_i - 2c_{n+1}f(p)_{d_i}}{|x_{n+1}|^2} & \frac{-a_i - 2d_{n+1}f(p)_{d_i}}{|x_{n+1}|^2} \end{pmatrix}.$$

The point $\hat{o}$ has coordinates

$$\psi(\hat{o}) = (1, 0, 0, 0, 0, \cdots, 0, 1, 0, 0, 0).$$

In that case,

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and $E_i = 0$ for $i = 2, \cdots, n$, so that, writing I_4 for the (4×4)-unit matrix, the differential of π at $\hat{o}$ becomes

$$d_{\psi(\hat{o})}(\varphi \circ \pi \circ \psi^{-1}) = \begin{pmatrix} I_4 & 0 & \cdots & 0 & -I_4 \\ 0 & I_4 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & I_4 & 0 \end{pmatrix}.$$

We can read off its kernel. Using $e_1, \cdots, e_4$ to denote the standard basis vectors of $\mathbb{R}^4$, we find

$$\ker d_{\psi(\hat{o})}(\varphi \circ \pi \circ \psi^{-1}) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} e_1 \\ 0 \\ \vdots \\ 0 \\ e_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ 0 \\ \vdots \\ 0 \\ e_2 \end{pmatrix}, \begin{pmatrix} e_3 \\ 0 \\ \vdots \\ 0 \\ e_3 \end{pmatrix}, \begin{pmatrix} e_4 \\ 0 \\ \vdots \\ 0 \\ e_4 \end{pmatrix} \right\}.$$

In terms of $\mathbb{H}^{n+1}$, this corresponds to

$$\ker d_{\hat{o}}\pi = \{(\lambda, 0, \cdots, 0, \lambda) \colon \lambda \in \mathbb{H}\}. \qquad \square$$

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