Department of Physics and Astronomy
Heidelberg University

Bachelor Thesis in Physics
submitted by

Paula Heim

born in München (Germany)

2023

Lee-Yang Theorem à la Lieb-Sokal

This Bachelor Thesis has been carried out by Paula Heim at the
Institute for Theoretical Physics in Heidelberg
under the supervision of
Prof. Dr. Manfred Salmhofer

Abstract. This thesis explores phase transitions in ferromagnetic systems using Ising-like models. The study focuses on the approach originally introduced by Lee and Yang in 1952 that relates phase transitions to the distribution of complex zeros of the partition function, and on their famous circle theorem which establishes that for certain models, these zeros are found on the unit circle. Lieb and Sokal extended this theorem to ferromagnets with arbitrary spins. The thesis is divided into two parts: the first provides a review of important concepts from thermodynamics and statistical mechanics and summarises the results of Lee and Yang, while the second presents an elaboration of Lieb and Sokal's proof of the generalised Lee-Yang theorem. The central idea of the proof for one-component ferromagnets involves using a differential operator to obtain the partition function from the partition function without interactions and showing that this operation preserves the property of complex zeros lying in the positive half-plane. The proof uses properties of complex polynomials and approximations with polynomials. Through a suitable transformation of variables, the two-component ferromagnet can be traced back to the one-component case. As a result, the Lee-Yang theorem for two-component ferromagnets with arbitrary spins can be deduced from the previously established results for the one-component ferromagnet.

Abstrakt. Diese Arbeit untersucht Phasenübergänge in ferromagnetischen Systemen mithilfe von Ising-ähnlichen Modellen. Die Untersuchung konzentriert sich auf den Ansatz, der ursprünglich von Lee und Yang im Jahr 1952 eingeführt wurde, und der Phasenübergänge mit der Verteilung komplexer Nullstellen der Zustandssumme in Verbindung bringt, sowie auf ihr berühmtes 'Circle-Theorem', das besagt, dass für bestimmte Modelle diese Nullstellen auf dem Einheitskreis liegen. Lieb und Sokal erweiterten dieses Theorem auf Ferromagneten mit beliebigen Spins. Die Arbeit ist in zwei Teile unterteilt: Der erste bietet eine Übersicht über wichtige Konzepte aus Thermodynamik und statistischer Mechanik und umfasst die Ergebnisse von Lee und Yang, während der zweite eine Ausarbeitung von Lieb und Sokals Beweis des verallgemeinerten Lee-Yang-Theorems präsentiert. Die zentrale Idee des Beweises für den einkomponentigen Ferromagneten besteht darin, einen Differentialoperator zu verwenden, um die Zustandssumme aus der Zustandssumme ohne Wechselwirkungen abzuleiten und zu zeigen, dass diese Operation die Eigenschaft der komplexen Nullstellen im positiven Halbraum zu liegen erhält. Der Beweis verwendet Eigenschaften komplexer Polynome und Approximationen mit Polynomen. Durch eine geeignete Variablentransformation kann der zweikomponentige Ferromagnet auf den einkomponentigen Fall reduziert werden. Dadurch kann das Lee-Yang-Theorem für zweikomponentige Ferromagneten mit beliebigen Spins aus den zuvor etablierten Ergebnissen für den einkomponentigen Ferromagneten abgeleitet werden.

Contents

1 Introduction

Phase transitions are intriguing phenomena that manifest in a wide spectrum of physical, chemical, and biological systems. These range from everyday occurrences such as the boiling of water and the melting of ice to Nobel prize winning experiments creating Bose-Einstein condensation, from the dynamics of bacterial populations to climate phenomena [MW19].

Phase transitions of magnetic systems are often modelled by Ising-like models which hold great significance for several reasons. Firstly, these models are relatively easy to analyse compared to more complex physical systems. This allows for a deeper understanding of the fundamental principles underlying phase transitions. Furthermore, Ising-like models serve as models for a wide range of real-world systems. They find applications in diverse areas such as ferromagnetism, biological systems and even social dynamics, demonstrating their versatility and relevance across disciplines [Isi+17; Lip22; Sta08; LB99].

One remarkable feature in the theory of phase transitions is the phenomenon of universality. Universality refers to the emergence of common behaviour near the critical point of a phase transition, regardless of the microscopic details of the system. For instance, the Ising model exhibits the same behaviour near a critical point as some systems that show liquid-vapour transitions. This implies that, by studying Ising-like models, we can extract valuable information which applies to a broader class of physical systems [ZJ21].

A major contribution to the study of phase transitions in the Ising model was made by Lee and Yang in 1952 [LY52a; LY52b]. They demonstrated a fundamental connection between the occurrence of phase transitions and the distribution of complex zeros of the partition function in the complex fugacity plane. Moreover, they proved the famous circle theorem. This theorem establishes that the accumulation points of the complex zeros of the partition function for Ising-like models, when regarded as a function of the fugacity, lie on the unit circle in the complex plane.

The beauty and simplicity of the circle theorem have attracted attention and inspired a multitude of subsequent research [FR12; BDL05]. Among these are the works of Newman, who investigated a Lee-Yang theorem for Ising-like models with general even or odd spin distributions [New74], and Dunlop, who extended the original Lee-Yang theorem to certain multi-component models, specifically the Heisenberg and plane rotator models [Dun79]. Building on the ideas of Newman and Dunlop, Lieb and Sokal managed to prove a similar theorem about the zeros of the partition function of a one-component ferromagnet with arbitrary spins [LS81]. Their method could even be extended to yield such a theorem for a two-component ferromagnet. This general Lee-Yang theorem stated by Lieb and Sokal is the theorem that we investigate in this thesis. The Lieb-Sokal version of the Lee-Yang theorem treats the partition function as a function of the external magnetic field and therefore does not show that the zeros are found on the unit circle, but that they are not found in the right half-plane. For models with a spin-flip symmetry, this can be restated as a circle theorem by passing to the exponential of the complex magnetic field.

This thesis is divided in two parts. In the first part, we review some concepts from thermodynamics and statistical mechanics. This will be done for two reasons. Our first aim is to remind the reader of the fundamental relationship between the partition function and phase transitions. Secondly, we develop the concepts and vocabulary to accurately guide us through the further discussion.

The second part of this thesis focuses on stating and, more importantly, proving Lieb and Sokal's generalised version of the Lee-Yang theorem, following the proof given by Lieb and Sokal. The central idea of the proof is to obtain the partition function of the ferromagnet from the partition function of the model without interparticle interactions through the application of a differential operator, and to show that the property of a function of having its complex zeros in the right half-plane is preserved by the application of this differential operator. This is a result that can, with a manageable effort, be proven for a polynomial function and a polynomial differential operator. In a series of topological arguments, we show that this property still holds after an approximation with polynomials in a certain space of entire functions. Proving that the partition function can be described with these entire functions finishes the proof of the Lee-Yang theorem for the general one-component ferromagnet. Hence the partition function of a particular model has no zeros in the right half plane if this is the case for the model without interactions. For the two-component ferromagnet, a suitable transformation of variables leads back to the one-component ferromagnet case, which means that a similar Lee-Yang theorem for the two-component ferromagnet can easily be deduced from the previous results.

2 Phase transitions, Lee-Yang approach and circle theorem

2.1 Phase transitions in thermodynamics

Thermodynamics provides a framework for understanding and predicting the behaviour of macroscopic systems. One intriguing aspect of thermodynamics is the study of phase transitions. Phase transitions are characterised by sudden changes in macroscopic observables, such as pressure or density. We will link the appearance of phase transitions to singularities in the equations of state. In this section, we introduce the necessary terminology to motivate and make precise this notion of a phase transition. This section largely follows Chapters 4.1, 5.1 and 5.2 in [Oli17], the paragraph about natural variables follows pp. 37 in [Cal85].

A *thermodynamic system* can be any body of matter or radiation, usually consisting of so many constituents that a description through many-body dynamics is neither possible nor interesting. Instead, thermodynamics focuses on the study of certain averaged or *macroscopic quantities*. The *state* of a thermodynamic system is described by several macroscopic quantities. Depending on the system, these may include its volume V, its temperature T, its particle number N, its pressure P, its entropy S or its chemical potential μ. A system is in *thermodynamic equilibrium* if, without external influences, its state does not change over time. We will only consider equilibrium systems. The space of all possible thermodynamic states is the space spanned by all possible values of the macroscopic quantities of a system. Only few of them are independent whereas the others can be regarded as functions of the independent ones. Such a relation is called *equation of state*.

There are various *thermodynamic potentials* that can be used to describe the state of a system. Each thermodynamic potential comes with a set of natural variables. The natural variables are the independent variables that are sufficient to determine the value of the thermodynamic potential in a given state. When expressed in its natural variables, the thermodynamic potential allows the derivation of other thermodynamic quantities from it by taking derivatives. This is explained in more detail below. Usually, one takes some of the above listed quantities as natural variables.

The internal energy U is perhaps the most obvious thermodynamic potential. Its natural variables are the entropy S, the volume V and the particle number N. Depending on the situation under consideration, it may however be more convenient to work with other variables. The interdependence of the thermodynamic variables permits a change of variables through a Legendre transform, allowing us to obtain the free energy

$$F(V,T,N) = \inf_S \{U(S,V,N) - TS\}$$

from the internal energy $U(S,V,N)$, and likewise other thermodynamic potentials such as the enthalpy $H(S,P,N)$, the Gibbs free energy $G(T,P,N)$ and the grand potential $\Phi_G(T,V,\mu)$. The Legendre transform above is always well-defined when U is convex as a function of S and an analogous statement holds for other the Legendre transforms.

As discussed before, only a few of all imaginable thermodynamic variables $X_1,\cdots,X_n$ are sufficient to uniquely determine the state of a thermodynamic system. Any thermodynamic potential A can therefore be understood as a function of only those variables $A(X_1,\cdots,X_n)$. If they are chosen appropriately, the total differential $\mathrm{d}A$ can be written as

$$\mathrm{d}A = \sum_{i=1}^{n} Y_i \mathrm{d}X_i,$$

and $X_1,\cdots,X_n$ are called the *natural variables* of the potential A.

The differential $\mathrm{d}A$ is exact, that is, it is the total differential of a function Ψ with the property

$$\left(\frac{\partial \Psi}{\partial X_i}\right)_{j\neq i} = Y_i, \tag{2.1}$$

where the notation $\left(\frac{\partial \Psi}{\partial X_i}\right)_{j\neq i}$ means that the variables X_j for $j \neq i$ are held constant.

The relations (2.1) yield functional dependencies of various thermodynamic variables to others in terms of derivatives of the thermodynamic potentials with respect to their natural variables. These are the equations of state.

A system that is completely uniform with regard to its thermodynamic properties constitutes a thermodynamic *phase*. The phases are open connected regions in the space of thermodynamic states, and within that region, the dependence of the thermodynamic variables on others are smooth. On the other hand, a *phase transition* occurs whenever there are singularities in the equations of state.

The study of phase transitions in thermodynamics thus requires the knowledge of a thermodynamic potential and the investigation of its analytic properties. In the next paragraph, we will encounter a method to derive the thermodynamic potential from the microscopic composition of a thermodynamic system.

2.2 Phase transitions in statistical mechanics

2.2.1 The approach of statistical mechanics

The central idea of statistical mechanics is to derive macroscopic properties and thermodynamic quantities from the underlying microscopic dynamics by studying the statistical behaviour of ensembles of particles. Its importance lies not only in the powerful tools it offers, but also in the rigorous foundation it lays for thermodynamics. In this section, we review fundamental concepts of statistical mechanics and how they relate to thermodynamics, in particular to phase transitions. We restrict the discussion to equilibrium systems.

Equilibrium states correspond to sets of systems in various phase space configurations that are compatible with certain macroscopic conditions, called *ensembles*. The ensembles can equivalently be characterised by probability distributions attributing a probability to each phase space configuration. The distributions are parameterised by thermodynamic variables. We will sometimes use the terms 'ensemble' and 'distribution' interchangeably.

As systems in nature obey the laws of quantum mechanics, it would in some situations be more correct to start with quantum statistical mechanics and the corresponding ensembles and then obtain classical statistical mechanics as a limit. This approach is in fact possible and it is outlined for example in Section 1.3.3 in [Rue07]. However, adopting this view would not provide any conceptual or technical benefit for the purpose of this thesis and we therefore limit the discussion to classical statistical mechanics.

The systems under consideration contain a very large number of particles. Recall that in one litre of air at room temperature alone there are $N = \mathcal{O}(10^{22})$ particles and molecules. Not only would it be impossible to analyse the dynamics of systems of such complexity in detail, it would also not be fruitful and interesting. In large systems, one is thus interested in collective behaviour and macroscopic physical quantities instead. Hence, we want to specify certain thermodynamic parameters and consider all possible microscopic configurations, or *microstates*, that are compatible with these. The probability with which the microstates are realised is encoded in the probability distributions that belong to the respective ensembles.

To an observable physical quantity we associate a function F and to an observation in a system its expectation value in the corresponding ensemble.

By $\mathbb{P}$ we denote the probability distribution on the phase space Ω corresponding to an ensemble. Then for any function F defined on Ω, it makes sense to ask for its mean value with respect to the probability distribution $\mathbb{P}$,

$$\langle F \rangle_{\mathbb{P}} = \int_{\Omega} F(w) \mathrm{d}\mathbb{P}(w).$$

We will later see that in this mathematical framework, however large, no system of finite particle size can truly account for the thermodynamic behaviour described in Section 2.1. It will therefore be necessary to consider a limit of infinite volume and particle number, the so-called *thermodynamic limit*. Nevertheless, below we first introduce ensembles for finite systems and explain how they relate to thermodynamic quantities, and then discuss the thermodynamic limit.

2.2.2 Thermodynamic states and the partition functions

We repeat the derivation of the microcanonical, the canonical and the grand canonical ensemble and their key properties. Unless indicated otherwise, we essentially follow part 1 of [Min00].

The microcanonical ensemble To derive the probability distributions corresponding to ensembles, we start from a microscopic description of a many-particle system. From a microscopic point of view, a thermodynamic system consists of a large number of individual, possibly interacting particles. Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain of volume $V = |\Lambda|$ where $d \in \mathbb{N}$ is the dimension of the system under consideration, and consider a system of N identical point particles moving inside Λ. Each particle i is characterised by a position q_i and a momentum p_i. We introduce $Q := (q_1, \cdots, q_N)$ and $P := (p_1, \cdots, p_N)$. The tuple (Q, P) is an element of the *phase space*

$$\Omega_{\Lambda,N} = \left(\Lambda \times \mathbb{R}^d\right)^N \subset \mathbb{R}^{2Nd}.$$

The Hamiltonian is one of the main characteristics of the system. It depends on the particle number N and the admissible domain Λ, and it usually has the form

$$H^{\Phi}_{\Lambda,N} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{X \subset \Lambda^N} \Phi(X),$$

where Φ is the interaction potential accounting for both the interactions between the particles and between a particle and an external field. For the existence of the thermodynamic limit, the potential Φ needs to be sufficiently short-ranged whilst also preventing the particles to collapse into a finite region of space. A more detailed discussion follows in Section 2.2.4.

The microscopic dynamics of the system is determined by the Hamilton equations of motion,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Writing $Q_t := (q_1(t), \cdots, q_N(t))$, $P_t := (p_1(t), \cdots, p_N(t))$, the dynamics of the system can be viewed as collection of maps $\{S_t\}_{t\in\mathbb{R}}$, where

$$S_t \colon \Omega_{\Lambda,N} \to \Omega_{\Lambda,N}, \qquad (Q_0, P_0) \mapsto (Q_t, P_t).$$

In the symplectic geometry picture, where the phase space is regarded as a symplectic manifold, this is also known as the Hamiltonian flow. By the Liouville theorem (Theorem 3.4 in [Bri22]), the dynamics given by the maps $\{S_t\}_t$ preserves the phase space volume, that is,

$$\operatorname{Vol} S_t(B) = \operatorname{Vol} B \quad \text{for all } B \subset \Omega_{\Lambda,N} \text{ and } t \in \mathbb{R},$$

and since we assume that the Hamiltonian is time-invariant, it also preserves the energy,

$$H^{\Phi}_{\Lambda,N}(S_t(Q_0, P_0)) = H^{\Phi}_{\Lambda,N}(Q_0, P_0) \quad \text{for all } t \in \mathbb{R}.$$

Each level surface of the energy

$$\Omega_{\Lambda,N,E} = \{(Q,P) \in \Omega_{\Lambda,N} \colon H_{\Lambda,N}(Q,P) = E\} \tag{2.2}$$

is thus mapped by S_t to itself,

$$S_t \Omega_{\Lambda,N,E} = \Omega_{\Lambda,N,E}.$$

Hence the dynamics on each surface can be considered separately which motivates the definition of the following measure.

Definition 2.1. The *microcanonical measure* $\lambda_{\Lambda,N,E}$ is defined on an energy level set $\Omega_{\Lambda,N,E}$ by

$$\lambda_{\Lambda,N,E}(B) := \lim_{\Delta E \to 0} \frac{\mathrm{Vol}(\Delta B)}{\Delta E},$$

where $B \subset \Omega_{\Lambda,N,E}$ and ΔB is formed by segments of normals to $\Omega_{\Lambda,N,E}$ that begin in B and extend to $\Omega_{\Lambda,N,E+\Delta E}$.

A proof for the existence of the limit whenever $\Omega_{\Lambda,N,E}$ is a regular submanifold of $\Lambda^{Nd} \times \mathbb{R}^{Nd}$ can be found in [GS64], pp. 220.

The microcanonical measure can be interpreted as measuring the density of states on the energy level surfaces. The precise form of $\lambda_{\Lambda,N,E}$ depends on Λ, N and E. Due to the preservation of energy and phase space volume, the measure remains invariant under the microscopic dynamics.

In the spirit of understanding thermodynamics, our focus shifts from the description of individual microstates to the specification of macroscopic quantities that characterise the system. Instead of considering the precise configurations (Q,P) of the particles of the system, we are interested in macroscopic observables. Importantly, for a given macroscopic state, there can be a multitude of microscopic configurations compatible with it in the sense that they yield the same values for the given macroscopic quantities.

A collection of configurations compatible with certain macroscopic constraints is called an ensemble. The microstates of an ensemble are distributed in $\Omega_{\Lambda,N}$ according to some density function $\rho(Q,P)$. When normalised to one,

$$\rho(Q,P)\prod_{i=1}^{N} \mathrm{d}^d q_i \mathrm{d}^d p_i$$

yields a probability measure on phase space. This implies that an ensemble can equivalently be described by some probability distribution $\mathbb{P}$ on $\Omega_{\Lambda,N}$.

The main postulate in statistical physics states that, given a macroscopic state, each microscopic configuration compatible with that state is equally likely to be realised by the system (pp. 127 in [Hua63]). The distribution must be invariant under time evolution, that is,

$$\mathbb{P}(B) = \mathbb{P}(S_t^{-1} B)$$

for all $B \subset \Omega_{\Lambda,N}$, because we are considering equilibrium systems. Together with the main postulate of statistical mechanics, we can now define the first ensemble by the following distribution.

Definition 2.2. The *microcanonical distribution* $\mathbb{P}^{\text{microcan.}}_{\Lambda,N,E}$ is the distribution obtained from normalising the microcanonical measure. It is given by

$$\mathbb{P}^{\text{microcan.}}_{\Lambda,N,E}(B) = \frac{\lambda_{\Lambda,N,E}(B)}{\lambda_{\Lambda,N,E}(\Omega_{\Lambda,N,E})}.$$

Up to a factor of $\lambda_{\Lambda,N,E}(\Omega_{\Lambda,N,E})^{-1}$, the measure is the same as the microcanonical measure. In particular, this implies that it is invariant under time evolution.

The canonical ensemble

In the microcanonical distribution, the energy of the system must be held constant. However, this constraint is often inconvenient in both theoretical considerations and experimental setups. Instead, one might be interested in situations where a system is in contact with a heat bath. This keeps its temperature fixed while allowing energy fluctuations between the system and the heat bath. The total of the system and the heat bath can now be described in the microcanonical ensemble. The probability distribution of the microstates of the system is obtained by integrating the microcanonical distribution over the parameters that belong to the heat reservoir. In other words, it is a marginal distribution (Chapter 7.1 in [Hua63], Chapters 4.7.1 and 6.14.1 in [FV18]).

We introduce the *inverse temperature* $\beta = \frac{1}{k_B T}$, where T is the temperature and Boltzmann's constant k_B is introduced so that β is measured in units of inverse energy. The variable β is more convenient to work with than T.

We can now define another ensemble, the *canonical ensemble*. It is characterised by the following probability distribution.

Definition 2.3. The *canonical distribution* $\mathbb{P}^{\text{can.}}_{\Lambda,N,\beta}$ is defined as

$$\mathbb{P}^{\text{can.}}_{\Lambda,N,\beta}(Q,P) = \frac{1}{\mathcal{Z}_{\Lambda,N,\beta}} \frac{1}{h^{Nd}} e^{-\beta H^{\Phi}_{\Lambda,N}(Q,P)}. \tag{2.3}$$

The normalisation factor $\mathcal{Z}_{\Lambda,N,\beta}$ is the *partition function*,

$$\mathcal{Z}_{\Lambda,N,\beta} = \frac{1}{h^{Nd}} \int_{\Omega_{\Lambda,N}} e^{-\beta H^{\Phi}_{\Lambda,N}(Q,P)} \prod_{i=1}^{N} \mathrm{d}^d q_i \mathrm{d}^d p_i. \tag{2.4}$$

The constant prefactor has no physical relevance and could also have been omitted. However, there are good reasons to include it. It is convenient to work with a dimensionless partition function because then, the distribution (2.3) can be thought of as a probability distribution. This is achieved by including a factor $\frac{1}{h^{Nd}}$ where h has the units of an action. The magnitude of h is irrelevant for our discussion.

Taking into account the invariance of both the Hamiltonian and the Lebesque measure $\prod_{i=1}^{N} \mathrm{d}^d q_i \mathrm{d}^d p_i$ with respect to the time evolution of the system, is is clear that the distributions (2.3) are also invariant under the dynamics.

It can be shown that, if the thermodynamic limit exists, the distribution (2.3) is equivalent to the microcanonical distribution in the sense that in the thermodynamic limit, the resulting distributions are the same everywhere except possibly at the points of phase transitions (pp. 145 in [Hua63], Chapter 4 in [Min00]).

The grand canonical ensemble

So far, the assumption has been made that the number of particles is always precisely known. This, however, is in most experimental situations not realistic, so that we introduce yet another distribution, the *grand canonical distribution*. When we let the system exchange particles with its environment, it makes sense to introduce the *chemical potential* μ as the change in internal energy that a change in particle number at constant volume and entropy causes. A similar consideration as before is used where the system is imagined as being in contact with a heat and particle reservoir and its probability distribution is again obtained as a marginal distribution from the microcanonical description of the total of system and heat bath. A certain value for the chemical potential replaces the fixed particle number N as macroscopic condition in the grand canonical ensemble.

The N-particle phase space $\Omega_{\Lambda,N}$ can no longer account for systems with variable particle number. A new phase space must therefore be introduced. Let $\lambda_\Lambda^{(N)} = \frac{1}{h^{Nd}} \prod_{i=1}^{N} \mathrm{d}^d q_i \mathrm{d}^d p_i$ be the dimensionless Lebesgue measure on $\Omega_{\Lambda,N}$. Then, the phase space of a system of variable particle number can be defined as the union of the N-particle phase spaces,

$$\Omega_\Lambda := \bigcup_{N\in\mathbb{N}_0} \Omega_\Lambda^{(N)},$$

where $\Omega_\Lambda^{(0)} = \emptyset$ can be identified with the vacuum.

We introduce a measure λ_Λ on Ω_Λ using the measures $\lambda_\Lambda^{(N)}$ in the following way,

$$\lambda_\Lambda(B^{(0)} \cup B^{(1)} \cup \cdots \cup B^{(N)} \cup \cdots) := \lambda_\Lambda^{(0)}(B^{(0)}) + \lambda_\Lambda^{(1)}(B^{(1)}) + \cdots + \lambda_\Lambda^{(N)}(B^{(N)}) + \cdots,$$

where $B^{(N)} \subset \Omega_\Lambda^{(N)}$ and we assume $\lambda_\Lambda(\emptyset) = 1$.

If the system is in a microstate $c = (q_1, \cdots, q_N) \in \Omega_\Lambda$, then its energy is taken to be $H_\Lambda^\Phi(c) = H_{\Lambda,N}^{\Phi^{(N)}}((q_1, \cdots, q_N))$ where $\Phi^{(N)}$ is the interaction potential that belongs to the N-particle Hamiltonian $H_{\Lambda,N}$, and Φ contains all potentials $\Phi^{(N)}$. It is assumed that $H_\Lambda^\Phi(\emptyset) = 0$.

With these preliminaries, we can define the grand canonical ensemble which is characterised by the following probability distribution.

Definition 2.4. Let $c \in \Omega_\Lambda$ and $N(c)$ is the number of particles in the configuration c. We define the *grand canonical distribution* as the following probability distribution on the phase space Ω_Λ.

$$\mathbb{P}_{\Lambda,N,\beta}^{\text{grandcan.}}(c) = \frac{1}{\mathcal{Z}_{\Lambda,\beta,\mu}^G} e^{-\beta H_\Lambda^\Phi(c) + \beta\mu N(c)}. \tag{2.5}$$

The normalisation factor $\mathcal{Z}^G_{\Lambda,\beta,\mu}$ is the *grand partition function*

$$\begin{aligned}\mathcal{Z}^G_{\Lambda,\beta,\mu} &= \int_{\Omega_\Lambda} e^{-\beta(H^\Phi_\Lambda(c)-\mu N(c))}\mathrm{d}\lambda_\Lambda(c)\\ &= 1+\sum_{N=1}^{\infty} e^{\beta\mu N}\frac{1}{h^{Nd}}\int_{\Omega_\Lambda} e^{-\beta H^{\Phi(N)}_{\Lambda,N}(c)}\mathrm{d}\lambda^{(N)}_\Lambda(c) = 1+\sum_{N=1}^{\infty} z^N \mathcal{Z}_{\Lambda,N,\beta}\end{aligned}$$

where the *fugacity* $z := e^{\beta\mu}$ was introduced and $\mathcal{Z}_{\Lambda,N,\beta}$ is the corresponding canonical partition function.

As in the case of the canonical and the microcanonical distribution, it can be shown that, if the thermodynamic limit exists, the canonical and the grand canonical distribution are equivalent in the sense that in the thermodynamic limit, the resulting distributions are the same everywhere except possibly at the points of phase transitions (pp. 157 in [Hua63], Chapter 4 in [Min00]).

2.2.3 Partition functions and thermodynamic quantities

In this section, we establish the link between statistical physics and thermodynamics. It essentially relies on a suitable definition of entropy. We denote $V = |\Lambda|$ and follow Chapters 6.3, 7.1 and 7.3 in [Hua63].

For the microcanonical ensemble, we can take the *Boltzmann entropy*, defined by

$$S(\Lambda, N, E) := k_B \log \lambda_{\Lambda,N,E}(\Omega_{\Lambda,N,E}), \tag{2.6}$$

where k_B is Boltzmann's constant and will be set to 1 from now on. The Boltzmann entropy (2.6) has the properties of the entropy in thermodynamics. These are that S is an extensive quantity, meaning that S increases when adding several systems together, and that, in accordance with the second law of thermodynamics, the entropy of an isolated system never decreases (this is established in Chapter 6.2 of [Hua63] and pp. 27 in [FV18]).

Motivated by the corresponding Maxwell relation in thermodynamics (p. 50 in [Oli17]), the temperature T of a system can be defined by

$$\frac{1}{T} := \frac{\partial S(\Lambda, N, E)}{\partial E}.$$

It can be shown that with this definition, the temperature is precisely the parameter that is equal in two otherwise isolated systems that are in equilibrium (see for example [Hua63], p. 133).

Other thermodynamic quantities can be defined analogously to the corresponding identities in thermodynamics,

$$P = T\left(\frac{\partial S}{\partial V}\right)_E, \quad \mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V, \quad U(S,V,N) = \int \mathrm{d}E, \quad F = U - TS, \quad c_V = \left(\frac{\partial U}{\partial T}\right)_V.$$

In general, these quantities are dependent on the parameters of the microcanonical distribution Λ, N, E, but for better readability, we omit carrying along the arguments for the most part.

Considering the canonical ensemble, suppose we coarse-grain phase space in a way that all states in a phase space cell have energies between E and $E + \Delta E$. We associate the value E with the microstates in such a phase space cell. As the energy of a system usually varies continuously, the procedure of coarse-graining is necessary to make the set of equal energy phase space cells countable. This has the advantage that we can express the canonical partition function as a sum over different energy level sets,

$$\mathcal{Z}_{\Lambda,\beta,N} = \sum_n e^{-\beta E_n} \lambda_{\Lambda,N,E_n}(\Omega_{\Lambda,N,E_n}) = \sum_n e^{-\beta E_n + S(\Lambda,N,E_n)}$$

where $\{E_n : n \in \mathbb{N}\}$ are the possibly discretised energies.

A comparison with the definition of the free energy from thermodynamics,

$$F(T,V,N) = \inf_S \{U(S,V,N) - TS\}$$

suggests the identification

$$\mathcal{Z}_{\Lambda,N,\beta} = e^{-\beta F(\beta,\Lambda,N)}.$$

Our argument is far from a mathematically rigorous derivation. We claim that this identification works and refer to the literature (pp. 144 in [Hua63] or pp. 31 in [FV18]) for a more thorough treatment.

Other thermodynamic quantities can again be derived from the corresponding identities in thermodynamics,

$$P = -\left(\frac{\partial F}{\partial T}\right)_V, \quad S = -\left(\frac{\partial F}{\partial T}\right)_V, \quad U = F + TS.$$

These definitions are compatible with the quantities that were defined before.

In the grand canonical ensemble, a similar approach can be employed to derive from

$$\mathcal{Z}^G_{\Lambda,\beta,\mu} = \sum_{N,E_n} e^{\beta\mu N - \beta U} \lambda_{\Lambda,N,E_n}(\Omega_{\Lambda,N,E_n}) = \sum_{N,E_n} e^{\beta\mu N - \beta U + S(\Lambda,N,E)}$$

and the following relation from thermodynamics,

$$\Phi_G(T,V,\mu) = \inf_{N,S} \{U(S,V,N) - TS - \mu N\} = -PV,$$

where the second equation results from the Euler relation for homogeneous systems (p. 8 in [FV18]), the identification

$$P = \frac{1}{\beta V} \log \mathcal{Z}^G_{\Lambda,\beta,\mu} = \frac{1}{\beta V} \log \left(1 + \sum_{N=1}^{\infty} z^N \mathcal{Z}_{\Lambda,N,\beta}\right). \tag{2.7}$$

A more rigorous treatment can be found in the literature (pp. 33 of [FV18] or pp. 149 in [Hua63]).

Other thermodynamic quantities can be derived from the corresponding identities in thermodynamics,

$$U = \frac{\partial}{\partial\beta}(\beta\Phi_G), \quad c_V = -\left(\frac{\partial U}{\partial T}\right)_V, \quad S = -\int_0^T \mathrm{d}T \frac{c_V}{T}, \quad F = U - TS.$$

Anything that was already defined is compatible with these definitions.

From the identifications we gave it is obvious that in statistical mechanics, as in thermodynamics, thermodynamic quantities are related to the characteristic potential and its derivatives. We therefore make our previous notion of a phase transition more precise in order to base the following discussion on a rigorous foundation.

Definition 2.5. We speak of a *phase transition* whenever the characteristic potential as a function of the thermodynamic parameters exhibits a lack of smoothness.

2.2.4 The thermodynamic limit

For a finite system, the pressure (2.7) is smooth as a function of β, V and z because the grand partition function is positive and the logarithm is an analytic function on the positive real numbers. This is also true of the other thermodynamic quantities. A finite system therefore never shows a phase transition. Furthermore, relative fluctuations in finite systems are non-negligible, leading to non-equivalence of ensembles. This demonstrates that finite systems are not sufficient to obtain thermodynamic behaviour.

When describing thermodynamics, we identify equilibrium states with the infinite limits of the ensembles. These limits will not always exist, and it is therefore important to verify that this is the case before applying the methods of statistical mechanics. As we are interested in systems with thermodynamic behaviour, we expect that the equilibrium states describe a situation where

(a) each subsystem has negligible interaction energy with the subsystems that are far away,

(b) in a bounded region of space, the number of subsystems remains finite.

This implies that interactions must be weak whenever subsystems are far away from each other, and that they prevent the collapse of an infinite number of subsystems into a bounded region of space.

Depending on the system, mathematically formalising the two conditions can be a rather intricate task. A comprehensive study of sufficient conditions can be found in Chapters 2 and 3 in [Rue07] or in Chapter I.A in [Len73]. In this section, we focus on lattice systems with pair interactions described in the grand canonical ensemble only since these include the ferromagnetic models that we are interested in. Our arguments follow the discussion in Chapter 2 of [Rue07].

In the grand canonical ensemble, taking the thermodynamic limit involves letting the volume become arbitrarily large, and we first need to specify how to do that.

Definition 2.6. Let $(\Lambda_n)_{n\in\mathbb{N}} \subset \mathbb{Z}^d$ a sequence of bounded sets. We define $V(\Lambda_n)$ to denote the cardinality of Λ_n and $V_h(\Lambda_n)$ the cardinality of the set of points with distance at most h to the boundary of Λ_n. We say that $(\Lambda_n)_{n\in\mathbb{N}} \subset \mathbb{Z}^d$ *tends to infinity in the sense of Van Hove* if

$$\lim_{n\to\infty} V(\Lambda_n) = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{V_h(\Lambda_n)}{V(\Lambda_n)} = 0.$$

We write $\Lambda \overset{\text{V.H.}}{\to} \infty$ when considering the limit $n \to \infty$ for any sequence $(\Lambda_n)_{n\in\mathbb{N}} \subset \mathbb{Z}^d$ that tends to infinity in the sense of Van Hove.

The notation $\Lambda \overset{\text{V.H.}}{\to} \infty$ is justified because the conditions of the previous definition guarantee that, if the limit of a quantity that depends on the volume exists, it is independent of the sequence chosen (see Chapter 2.1 in [Rue07] and pp. 83 in [FV18]).

In the context of the next theorem, let an interaction be a translation-invariant function

$$\Phi\colon \{X \subset \mathbb{Z}^d \colon X \text{ finite}\} \to \mathbb{R}, \qquad \Phi(X+a) = \Phi(X) \ \text{ for all } a \in \mathbb{Z}^d,$$

satisfying $\Phi(\emptyset) = 0$. We define the map $\|\cdot\|$ on the space of interactions

$$\|\Phi\| = \sum_{X\colon 0\in X} \frac{|\Phi(X)|}{|X|}.$$

On the subspace of interactions $\mathscr{B} = \{\Phi\colon \|\Phi\| < \infty\}$, the map $\|\cdot\|$ is a norm, and equipped with this norm, $\mathscr{B}$ is a Banach space.

If we denote $\mathscr{B}^k = \{\Phi\colon \Phi(X) = 0 \text{ for } |X| \neq k\}$ the space of k-particle interactions, then *pair interactions* are of the form $\Phi = (\Phi^1, \Phi^2) \in \mathscr{B}^1 \oplus \mathscr{B}^2$. The pressure (2.7), from which other thermodynamic variables can be derived in the grand canonical ensemble, depends on the interaction through the Hamiltonian $H_\Lambda^\Phi(q_i, p_i) = \sum_i \frac{p_i^2}{2m} + \Phi((q_i)_i)$ in the partition function. Theorem 2.7 will indicate that the interaction plays a role in determining if the pressure is well-defined in the thermodynamic limit. To make the dependence of the pressure on the interaction clear, we write Φ as an argument of the pressure. With our previous considerations, this is

$$P_\Lambda(\Phi) = \frac{1}{\beta\,|\Lambda|} \log \mathcal{Z}^G_{\Lambda,\beta,\mu} = \frac{1}{\beta\,|\Lambda|} \log \int_{\Omega_\Lambda} e^{-\beta(H_\Lambda^\Phi(c) - \mu N(c))} \mathrm{d}\lambda_\Lambda(c), \tag{2.8}$$

where c is a configuration in the phase space Ω_Λ. In the previous sections, the pressure has always been denoted without the argument Φ which is only introduced for clarity here. In the following sections, we will drop the argument again, but, of course, the dependence remains valid.

The following theorem guarantees the existence of the thermodynamic limit of (2.8).

Theorem 2.7. *For all pair interactions $\Phi \in \mathscr{B}^1 \oplus \mathscr{B}^2$, let the pressure $P_\Lambda(\Phi)$ be defined by* (2.8)*. Then, in the thermodynamic limit, the pressure exists and is finite. It is given by*

$$P(\Phi) = \lim_{\Lambda \overset{V.H.}{\to} \infty} P_\Lambda(\Phi).$$

The proof of Theorem 2.7 involves deriving such a statement for all interactions $\Phi \in \mathscr{B}$ and then obtaining the thermodynamic limit for pair interactions as a special case of the general theorem. Giving this proof would go beyond the scope of this thesis, therefore we simply refer to the literature (pp. 23 in [Rue07]) instead. It is nonetheless interesting (but not surprising, as classical statistical mechanics must be a limit of quantum statistical mechanics) to note that the general theorem can be derived from the quantum case by expressing the classical statistical ensembles as limits of quantum statistical ensembles.

It can be shown (see for example Chapter 8 in [Min00], Chapters 2 and 3 in [Rue07], Chapter 1 in [Len73], parts of Chapters 6 and 7 in [Hua63] and pp. 121 in [Gal99]) that under the condition that the interaction is in $\mathscr{B}^1 \oplus \mathscr{B}^2$, when replacing a thermodynamic quantity $\mathscr{X}$ by its density per volume x_Λ,

$$x_\Lambda = \frac{\mathscr{X}}{|\Lambda|}$$

the limits of the densities exist in the thermodynamic limit. Hence

$$x = \lim_{\Lambda \overset{\text{V.H.}}{\to} \infty} x_\Lambda$$

is well-defined and describes the thermodynamic behaviour of the system.

All systems in nature are, of course, finite, so there is a legitimate question as to why we can observe phase transitions at all if they only occur in infinite systems. A possible answer, given by [FV18], pp. 47, is that real systems are quite large, thus they exhibit behaviour that is almost singular. In an experiment this will be indistinguishable from real singularities. In addition, we always make simplifications when modelling physical processes because the complexity of even the simplest situations, if described in full detail, would often be beyond our mathematical tools and computational capabilities. Taking the thermodynamic limit is therefore well justified in that it allows us to study phase transitions by investigating the analytical properties of the thermodynamic functions. This is not to say that finite systems are not per se interesting in the context of studying phase transitions. There is a well-developed theory of finite-size scaling which describes the deviation of the behaviour of finite systems from those of infinite systems due to finiteness [BK90].

2.3 Modelling phase transitions in ferromagnets

2.3.1 The Ising model and two variants

Ferromagnetism is the phenomenon of spontaneous alignment of magnetic moments in materials. Ferromagnetic phase transitions occur when ferromagnetic materials undergo sudden changes in their magnetic properties as external conditions vary. The simplest model for such phenomena is the Ising model. Proposed by Wilhelm Lenz in 1920 and investigated by his student Ernst Ising, it was the first significant approach to understand these transitions mathematically [Isi+17]. Ising's work [Isi25] provided an exact solution for the one-dimensional Ising model but the model failed to exhibit a phase transition. In 1936, Rudolf Peierls published a famous argument [Pei36] indicating that spontaneous symmetry breaking occurs at nonzero temperature in the Ising model in two or more dimensions, opening the door to further research into the model. It was not until eight years later that an exact solution of the two-dimensional Ising model was obtained by Lars Onsager, thereby determining the exact values of the thermodynamic variables at which the transition occurs [Ons44]. Peierls' argument was turned into a rigorous proof by Griffiths [Gri64] and, independently, by Dobrushin [Dob68]. The Ising model has gained significant popularity since then and found numerous applications beyond ferromagnetism, extending to diverse fields such as biological processes (Chapter 2.7 in [LB99]), economics [Lip22] and social behaviour [Sta08].

In this section, we formally introduce the original Ising model and two variants of it. The variants are slight generalisations of the original Ising model. The first variant no longer restricts the spin-spin interactions to nearest-neighbour interaction. This is the model for which Lee and Yang proved the circle theorem. The second variant is a further generalisation of the first variant in that the possible spin values are not restricted to $\{\pm 1\}$. This is the model for which a Lee-Yang theorem will be proven in the second part of this thesis. For the definition of the models, we follow [FR12].

The Ising model reproduces the behaviour of ferromagnetic materials by representing them as a collection of particles with inherent magnetic moments or spins. It can be formulated as a (weighted) graph, where spins represented by random variables ϕ_i are placed at the vertices, and edges are added between interacting spins (with edge weights accounting for different interaction strengths). It is often convenient to consider an embedding of the model in $\mathbb{R}^d$ such that the graph becomes a sublattice of $\mathbb{Z}^d$, where d is the dimension of the model. This embedding allows for an intuitive correspondence to experimentally realisable physical systems and introduces a well-defined notion of distance, enabling precise interpretations of range-dependent interactions. We will henceforth adopt this viewpoint.

Let therefore $\Lambda \subset \mathbb{Z}^d$ be a finite sublattice and associate to each site i a random variable ϕ_i, referred to as spin. We allow the spin to take arbitrary values in some set $\mathcal{A} \subset \mathbb{R}^N$. To avoid confusion, we emphasise that N in this context denotes the number of spin components, not the particle number. From now on, N will consistently be used with this meaning. While it is common to consider $N = 1$ and refer to it as a *one-component ferromagnet*, the study of multi-component ferromagnets is equally intriguing. We focus on the case $N = 1$ here and return to multi-component ferromagnets in Section 3.5. Let ν_i be a probability measure on $\mathcal{A}$ that captures the a-priori distribution of the random variable ϕ_i. Note that the set $\mathcal{A}$ must be chosen such that, with an appropriate σ-algebra $\mathscr{F}$, the triple $(\mathcal{A}, \mathscr{F}, \nu_i)$ is a probability space. The space of spin configurations is

$$\mathcal{A}^\Lambda = \{\phi_\Lambda := (\phi_x)_{x\in\Lambda} \colon \phi_x \in \Omega \quad \text{for all } x \in \Lambda\}\,.$$

The interactions between the spins can be described by a potential Φ which associates with every subset $X \subset \Lambda$ an interaction energy $\Phi(X)$ of all spins in X. We assume that Φ is a pair interaction of the form

$$\Phi(X) = \sum_{(i,j)\in X^2} J_{ij}\phi_i\phi_j + \sum_{i\in X} h_i\phi_i, \tag{2.9}$$

where $J_{ij} = J_{ji}$ is the interaction coefficient and h_i is the external magnetic field at site i. For a ferromagnetic interaction, one takes $J_{ij} \geq 0$. The Hamiltonian of the system is then

$$H_\Lambda^\Phi = -\left(\sum_{(i,j)\in\Lambda^2} J_{ij}\phi_i\phi_j + \sum_{i\in\Lambda} h_i\phi_i \right).$$

For the existence of the thermodynamic limit, according to the discussion in Section 2.2.4 and specifically Theorem 2.7, it is sufficient to show that Φ is translation-invariant and that

$$\|\Phi\| = \sum_{X\,:\,0\in X} \frac{|\Phi(X)|}{|X|} = \sum_{X\,:\,0\in X} |J_{i0}\phi_i\phi_0| + |h_0\phi_0| < \infty. \tag{2.10}$$

Translation-invariance is certainly given when J_{ij} only depends on $|i-j|$, and when, in addition, the magnetic field is homogeneous, that is, $h_i = h$ everywhere. The condition (2.10) is fulfilled whenever each spin only interacts with finitely many other spins or when the strength of the interaction decreases sufficiently fast with increasing distance.

If these conditions are not fulfilled, then the thermodynamic limit can still exist, or it could exist in a different (weaker) sense, but, with the tools at hand, we cannot make any statement about it. However, in a later section in Theorem 2.9, we address this issue again for the models of interest.

In the original Ising model, the interaction is restricted to nearest-neighbour interactions which means that $J_{ij} = J \neq 0$ if and only if i, j are nearest-neighbours. The spin variables ϕ_i can take values of ± 1 with equal probability,

$$\mathrm{d}\nu(\phi_i) = \frac{1}{2}\left(\delta(\phi_i - 1) + \delta(\phi_i + 1)\right). \tag{2.11}$$

Definition 2.8. We now define two slightly generalised versions of this model.

(a) In this model, we assume that the spins are distributed according to (2.11) and the magnetic field is homogeneous, but we allow arbitrary interaction coefficients J_{ij}.

(b) In this model, the spin distributions ν_i can be any probability measures on $\mathcal{A}$ and we allow arbitrary interaction coefficients J_{ij} and magnetic fields h_i.

For all these models, we say that they are *ferromagnetic* whenever spin alignment lowers the energy which is precisely the case when $J_{ij} \geq 0$ for all i, j.

2.3.2 The Ising model in statistical mechanics

At first sight, it may not seem obvious that the Ising model can be adequately described within the framework of thermodynamics and statistical physics, since conventional thermodynamic quantities such as pressure, density and specific heat have no obvious interpretation. Lee and Yang contributed to establishing a solid foundation for analysing Ising-like models with the tools of statistical physics. They demonstrated the mathematical equivalence between models of type 2.8(a) and a lattice gas [LY52b]. Recall that a lattice gas involves sites on a lattice which can be either occupied or unoccupied. Through a simple transformation of variables, the occupied sites are identified with (+1)-spins, and the unoccupied sites with (−1)-spins. This transformation establishes a direct correspondence for various key properties. Specifically, the following table summarises the quantities are equivalent between the two models. Note that some correspondences are expressed as functions of the listed quantities.

Ising model	Lattice gas
Number of spins M	Volume V
Number of down spins	Number of atoms
Magnetisation $\frac{2}{1-m}$	Specific volume v
Free energy density $-f-h$	Pressure P
Fugacity $y=e^{-2\beta h}$	Fugacity $z=e^{\beta\mu}$

More generally, also the two variants 2.8(a) and (b) can be described within the previously established framework by replacing the parameter μ that determines the grand canonical distribution with the magnetic field h, although for the models 2.8(b), there is in general no correspondence to a lattice gas. The grand canonical distribution takes on the form

$$\mathbb{P}^{\text{grandcan.}}_{\Lambda,\beta,h}(c)=\frac{1}{\mathcal{Z}^G_{\Lambda,\beta,h}}e^{-\beta H^{\Phi}_{\Lambda}(c)}, \tag{2.12}$$

where $c\in\mathcal{A}^{\Lambda}$ is a spin configuration. Note that the magnetic field has been absorbed into the definition of the Hamiltonian, unlike the chemical potential in (2.5). The partition function becomes

$$\mathcal{Z}^G_{\beta,\Lambda,h}=\int\limits_{\mathcal{A}^{\Lambda}}e^{-\beta H^{\Phi}_{\Lambda}(\phi_\Lambda)}\prod_{i\in\Lambda}\mathrm{d}\nu_i(\phi_i). \tag{2.13}$$

For the Ising model (and the generalisation 2.8(a)), where

$$\mathrm{d}\nu_i(\phi_i)=\frac{1}{2}\left(\delta(\phi_i-1)+\delta(\phi_i+1)\right)\qquad\text{for all } i,$$

the partition function can be expressed as a polynomial in y of degree M ([FV18] pp. 83).

For the Ising-like models of type 2.8(b) with more general spin distributions, the partition function may no longer be polynomial, but (2.13) can still be used to define the partition function.

We already discussed that $\sum\limits_{X:\,0\in X}\frac{|\Phi(X)|}{|X|}<\infty$ is a sufficient condition for the free energy density

$$f:=-\lim_{\Lambda\overset{\text{V.H.}}{\to}\infty}\frac{1}{\beta\,|\Lambda|}\log\mathcal{Z}_{\beta,\Lambda,h} \tag{2.14}$$

to exist and be finite whenever $\beta>0$, and we confirmed that the original Ising model and certain variants of it fulfils the condition. Technically speaking, it would be correct to use the term 'grand potential density' instead of free energy density in accordance to the terminology developed in Section 2.1, but as (2.14) is usually referred to as the free energy density, we save ourselves the linguistic pedantry and simply refer to it as the free energy density.

The physical quantities of interest are the 'pressure' P, the mean magnetisation m and its derivatives. Each spin carries a magnetic moment so that we define the mean magnetisation m_Λ of a finite system with volume Λ as

$$m_\Lambda = \langle m_\Lambda((\phi_i)_{i\in\Lambda})\rangle_{\mathbb{P}^{\text{grandcan.}}_{\Lambda,\beta,h}}, \qquad \text{where } m_\Lambda((\phi_i)_{i\in\Lambda}) = \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\phi_i. \tag{2.15}$$

Through the grand canonical distribution $\mathbb{P}^{\text{grandcan.}}_{\Lambda,\beta,h}$ it depends on Λ, β and h, but, for the sake of readability, we decided not to include arguments or indices that could make the dependence on β and h explicit. The pressure of a finite system P_Λ is a function of Λ, β and h is given by (2.7),

$$P_\Lambda := \frac{1}{\beta\,|\Lambda|}\log \mathcal{Z}^G_{\Lambda,\beta,h}. \tag{2.16}$$

Again, we refrain from carrying the arguments β, h. We know that a phase transition can only occur in the infinite volume limit

$$P := \lim_{\Lambda \overset{\text{V.H.}}{\to} \infty} P_\Lambda,$$

which is a function of β and h. The mean magnetisation in the thermodynamic limit is obtained as

$$m := \lim_{\Lambda \overset{\text{V.H.}}{\to} \infty} \langle m_\Lambda\rangle_{\mathbb{P}^{\text{grandcan.}}_{\Lambda,\beta,h}} = \frac{\partial P}{\partial h}, \tag{2.17}$$

which is also a function of β and h. It is not obvious that the limit and differentiation can be swapped, but this is in fact the case and we refer to our Theorem 2.9 for a proof sketch and to [FV18], pp. 89, for a complete proof. The first partial derivative of the magnetisation m by the external field h describes the magnetic susceptibility χ,

$$\chi = \frac{\partial m}{\partial h},$$

which, again, is a function of β and h. At a first-order phase transition, a discontinuity in m appears, whereas at a higher-order phase transition, one of its derivatives fails to be continuous.

2.3.3 Spontaneous symmetry breaking

In the case of $d = 1$ dimension, the free energy density (2.14) depends analytically on h in the thermodynamic limit for all finite values of β, thus precluding the possibility of a phase transition at non-zero temperature (see, for example, pp. 44 in [LB99], and of course [Isi25]). However, in $d \geq 2$ dimensions, this is no longer the case, and the ferromagnetic Ising model displays a phase transition. While an exact analytic description of the phase transition remains a challenging task in $d = 2$ (Chapter 7.1 in [Gal99]) and is a yet unsolved problem in $d = 3$ dimensions, a qualitative argument can be made for the occurrence of a phase transition from a disordered to an ordered phase. In the unordered phase and at zero magnetic field, models of the form 2.8(a) are symmetric under a global spin flip $(\phi_i)_{i\in\Lambda} \to (-\phi_i)_{i\in\Lambda}$. In the ordered phase, this symmetry is broken. Thus, the quantity whose change indicates a first order phase transition is the mean magnetisation m_Λ that was defined in (2.15). Note that it is possible for the magnetisation to be continuous, but one of its derivatives could be discontinuous. In that case, a phase transition of higher order occurs. In this section, we present a simple argument that indicates the presence of a symmetry breaking by investigating the grand canonical distribution of the model in the high- and low-temperature limits at zero external field. We follow pp. 45 in [FV18].

In the high-temperature limit ($T \to \infty$ or equivalently $\beta \to 0$) and for vanishing external magnetic field, the thermal energy of the spins dominates over the attractive interaction between neighbouring spins, causing the grand canonical distribution (2.12) to converge to a uniform distribution on the space of spin configurations $\mathcal{A}^\Lambda$. As a result, the magnetisation (2.15) becomes a sum of independent and identically distributed random variables ϕ_i with expectation value $\langle \phi_i \rangle_{\mathbb{P}^{\text{grandcan.}}_{\Lambda,\beta,h}} = 0$ for all i. Consequently,

$$\lim_{T\to\infty} m_\Lambda = 0$$

follows from the law of large numbers, and this must also hold in the thermodynamic limit.

Conversely, in the low-temperature limit, the distribution (2.12) at zero magnetic field favours states that minimise the Hamiltonian. For the ferromagnetic Ising model, the states of minimum energy are the ground states where all spins are aligned, leading to

$$\lim_{T\to 0} m_\Lambda \neq 0.$$

The global spin flip no longer preserves the symmetry.

The distinct behaviours for the system at high and at low temperatures demonstrate the existence of a phase transition that breaks the spin flip symmetry. However, no statement can be made about the temperature at which the phase transition occurs. We know that in $d = 1$ dimension, the transition happens at zero temperature whereas it occurs at a finite temperature in finite dimension. Moreover, the role of the magnetic field (that has been set to zero) needs to be examined. In the following, we present the findings of Lee and Yang whose approach addresses precisely the role of the magnetic field for phase transitions of the Ising model and conclusively proves that no phase transition is possible at nonzero magnetic field.

If we consider models of the form 2.8(b) with spin values distributed according to distributions $\{\nu_i\}_{i=1}^n$, the expected magnetisation in the high- and low-temperature limit may be different. It still holds that in the high-temperature limit, the spins are essentially independently distributed so that the mean magnetisation and the mean of the expectation values of the measures ν_i coincide,

$$\lim_{T\to\infty} m_\Lambda = \frac{1}{|\Lambda|}\sum_{i=1}^{n}\int_\Omega \mathrm{d}\nu_i.$$

As $T \to 0$, the system's grand canonical distribution will again favour its energy-minimising ground state(s). Without further information about the spin distributions $\{\nu_i\}_{i=1}^n$ it is impossible to make more precise statements about the occurence of spontaneous symmetry breaking. However, even if no symmetry breaking occurs, there can still be phase transitions of higher order. In the second part of this thesis, it is demonstrated that phase transitions in such systems can only occur at zero magnetic field, just like for the models of the form 2.8(a).

2.4 Lee-Yang approach and circle theorem

2.4.1 The central idea

We discussed in Section 2.2.3 that a characteristic potential (here the grand potential) is proportional to the logarithm of the partition function. Whenever this potential is an analytic function of the control parameters, the physical observables are also analytic, indicating that the system exists in a single phase. However, at the point of a phase transition, the characteristic potential, and therefore the logarithm of the partition function, exhibits a lack of smoothness. Lee and Yang realised that this lack of smoothness is related to zeros of the partition function [LY52a]. Their analysis applies to systems of the form 2.8(a), so we restrict the discussion of their results to those systems only.

Recall that the grand partition function of such a sytem with M spins can be written as a polynomial in the variable $y = e^{-2\beta h}$,

$$\mathcal{Z}^G_{\beta,\Lambda,h} = K\prod_{i=0}^{M}\left(1 - \frac{y}{y_i}\right). \tag{2.18}$$

The y_i are the zeros of the polynomial and $K \in \mathbb{R}$ is a for the analytic behaviour irrelevant constant. We established that $\mathcal{Z}^G_{\beta,\Lambda,h}$ is analytic whenever the number of particles and the volume are finite as a polynomial of finite degree in y. The control parameter is $y = e^{-2\beta h}$, because h is the quantity that can be varied experimentally. Obviously, y can only have nonnegative real values in a real world experiment. Consequently, $y_i \notin \mathbb{R}_+$ because the thermodynamic quantities of a finite system must depend smoothly on the parameters as such a system never exhibits a phase transition. It follows that $\mathcal{Z}^G_{\beta,\Lambda,h}$ is real and positive whenever $h \in \mathbb{R}$, so that the logarithm of the partition function and all its derivatives are smooth. This observation is consistent with the discussion in Section 2.2.4, and it shows once more that for the study of phase transitions, the thermodynamic limit must be investigated.

Lee and Yang investigated the information that the complex zeros of the partition function contain about the phase transitions. Their idea was to study the zeros of the partition function by employing complex values for the magnetic field h or equivalently complex values for the variable y as a computational tool. The polynomial (2.18) has its zeros somewhere in the complex y-plane but not on the physically accessible positive real y-axis. As the system size increases, the location of the zeros changes, and it is only in the thermodynamic limit that they might appear on the positive real y-axis. The relation between the logarithm of the partition function and the physical properties of the system suggests that a phase transition happens at the limit points of the zeros on the positive real y-axis. It was the accomplishment of Lee and Yang to prove this relation conclusively.

The distribution of the zeros of $\mathcal{Z}^G_{\Lambda,\beta,h}$ in the complex y-plane thus provides valuable insights into the occurrence and location the of phase transitions. By examining the accumulation points of the zeros in the thermodynamic limit, one can, in principle, identify the presence of a phase transition. Furthermore, the distribution enables the derivation of various quantities that describe the system's behaviour near the phase transition point. In the following we will give a comprehensive presentation of the exact results and statements on the distribution of zeros and their significance for the characterisation of phase transitions.

2.4.2 Zeros of the partition function and phase transitions

We state the precise relation between the zeros in the complex y-plane and the appearance of phase transitions as found by Lee and Yang [LY52a].

Theorem 2.9. *If in the complex y-plane a region containing a segment R of the positive real axis is free of zeros for all Λ, then the quantities*

$$\frac{1}{|\Lambda|}\log \mathcal{Z}^G_{\Lambda,\beta,h}, \quad \frac{\partial}{\partial \log y}\frac{1}{|\Lambda|}\log \mathcal{Z}^G_{\Lambda,\beta,h}, \quad \frac{\partial^2}{\partial \log y^2}\frac{1}{|\Lambda|}\log \mathcal{Z}^G_{\Lambda,\beta,h}, \quad \cdots$$

approach a limit that is analytic in y. Furthermore, the operations $\partial/\partial \log y$ and $\lim\limits_{|\Lambda|\to\infty}$ commute in R such that in the thermodynamic limit, the pressure P and the magnetisation m, given by

$$P = \lim_{|\Lambda|\to\infty}\frac{1}{\beta\,|\Lambda|}\log \mathcal{Z}^G_{\Lambda,\beta,h}, \quad m = 1 - 2\lim_{|\Lambda|\to\infty}\frac{\partial}{\partial \log y}\frac{1}{|\Lambda|}\log \mathcal{Z}^G_{\Lambda,\beta,h},$$

are analytic in the control parameter y. This is equivalent to the system being in a single phase.

The infinite volume limit considered by Lee and Yang is obtained by filling out the volume Λ with boxes that can be partitioned into cubes whilst assuming that the ratio $N/|\Lambda|$ remains bounded as the system size increases. Such a sequence tends to infinity in the sense of Van Hove because the ratio $\frac{\text{surface}}{\text{volume}}$ of a cube is proportional to $\frac{1}{\text{sidelength}}$ which tends to zero as the cubes become arbitrarily large. A similar statement holds for boxes composed by cubes.

The theorem holds regardless of whether the partition function is polynomial in the fugacity or not. A proof for the polynomial case can be found in Appendix II in [LY52a]. It consists of proving the statement for a circle C contained in R first by expressing $\frac{\log \mathcal{Z}_{\Lambda,\beta,h}}{|\Lambda|}$ as a power series. It is possible to confirm that the power series is absolutely convergent, this relies on the polynomial form of the partition function. Then, the summation and the limit $|\Lambda| \to \infty$ can be exchanged. The resulting power series is absolutely convergent in C and therefore represents a function that is analytic in the interior of C. This argument can be repeated until the circles fill all of R.[1]

Conversely, consider the opposite case in which a limit point y_1 of the complex zeros is found on the real y-plane as $|\Lambda| \to \infty$. The point y_1 divides the real axis into two segments R_1, R_2. In general, the location of y_1 is temperature-dependent. Within each of these segments, the pressure P and the magnetisation m are analytic, this follows from Theorem 2.9. At the point y_1, the pressure is continuous (this is proven in Theorem 1 in [LY52a]), but its derivative m has a discontinuity, indicating a phase transition at that point. We stress that the phase transition is a consequence of the variation of the external field h while the inverse temperature β remains constant. However, the value of $y = e^{-2\beta h}$ at which it occurs is in general temperature-dependent.

Lee and Yang made some more observations that relate different behaviours of the zeros in the complex y-plane to different physical scenarios, these are listed below.

- If, at a certain temperature T_c the zeros cease to close in onto the real axis at one of the points as the temperature exceeds T_c, then T_c is a critical temperature for the transition.
- In the case where there exist two such points y_1, y_2, and they merge at some temperature T_0, then there is a triple point at T_0.
- It is possible for the density to be continuous at y_1, but then one of its derivatives will in general be discontinuous. This behaviour typically occurs at the critical temperature but not at neighbouring temperatures. If it persists over an extended temperature range, there is a phase transition of higher order.

2.4.3 Circle theorem

In the preceding section, the profound connection between the zeros of the partition function and the occurrence of phase transition has been explored. From this relation it is evident that we can gain insights into the phase transition by studying the distribution of the zeros in the complex y-plane, where $y = e^{-2\beta h}$. However, determining this distribution analytically is a challenging and for many models not yet accomplished task. It was therefore remarkable that Lee and Yang found that the zeros in the complex y-plane of the partition function of ferromagnetic Ising models of the form 2.8(a) are on the unit circle [LY52b]. This implies that a phase transition can possibly occur at no more than one value of y, this is the intersection of the unit circle and the positive real y-axis. The regularity of the distribution of the zeros not only greatly restricts where a phase transition can occur, but it also allows thermodynamic properties to be calculated, as is shown below.

[1] We emphasise that the theorem is also valid for non-polynomial partition functions, this was just not proven by Lee and Yang. We briefly sketch a proof for the more general models 2.8(b) that are discussed in Section 3. For these models, we can build a proof upon the idea of Lee and Yang that was sketched above. Suppose that the partition function can be approximated by polynomials. We will see in the second part of this thesis that this is a reasonable assumption for the models under consideration. It is therefore analytic in R which implies that its Taylor series converges in all of R. For the approximating polynomials, the above theorem can be applied in the form in which Lee and Yang proved it. This yields a sequence of functions that are analytic in R. There are two limit-taking procedures now, the limit of the polynomials to the partition function and the infinite-volume limit of the pressure for these polynomial approximating partition functions. In order for the theorem to hold in the desired way, we need to exchange these two limits. But this is possible because both series are absolutely convergent in R.

Lee and Yang formulated the circle theorem for a lattice gas where it holds for attractive interactions. We state the theorem for the Ising-like models 2.8(a) with spin-spin interaction coefficients J_{ij} that are not restricted to nearest-neighbour interactions.

Theorem 2.10. *Under the condition that $J_{ij} \geq 0$ for all $(i,j) \in \Lambda^2$ (where $J_{ii} = 0$), the zeros of the partition function $\mathcal{Z}^G_{\Lambda,\beta,h}$ of a model of the form 2.8(a) lie on the unit circle in the complex y-plane.*

It is worth noting that this theorem holds irrespective of the range of interaction, the dimension of the lattice, its size and structure.

The proof of the theorem is based on the invariance of the polynomial partition function (2.18) under $y \to \frac{1}{y}$, which results from the symmetry of the models under a global spin flip with a simultaneous reversion of the magnetic field, $h \to -h$. The theorem is then a consequence of properties of complex polynomials with this specific symmetry. This also explains why it is difficult to find similar theorems for other models, because their partition functions need not have this symmetry or even be polynomial.

A significant implication can be drawn immediately from the circle theorem. While it does not indicate whether the zeros converge to a point on the positive real y-axis, if such an accumulation were to occur, it could only happen at $y = 1$ in the models 2.8(a), which is equivalent to $h = 0$. Consequently, the existence of a phase transition is restricted to zero external magnetic field.

It is also worth noting that studying the distribution of the zeros reveals more properties of the phase transition than merely its location. From the distribution of zeros on the unit circle, we can derive the grand potential and from it obtain thermodynamic quantities such as the pressure and the magnetisation. Although this is not further investigated or applied in the further course of this thesis, we present the ideas because they have many attractive applications to the study of phase transitions of systems for which a circle theorem holds (for example [MH77]).

We introduce a distribution density function $g(\theta)$ so that $Mg(\theta)\mathrm{d}\theta$ is equal to the number of zeros z between $e^{i\theta}$ and $e^{i(\theta+\mathrm{d}\theta)}$. The distribution function is symmetric under $\theta \to -\theta$ as a consequence of the already mentioned spin-flip symmetry with a reversal of the magnetic field.

Factorising the polynomial and taking the logarithm, one obtains the free energy density in the infinite volume limit for the Ising model,

$$f = h + \frac{1}{\beta} \int_0^{2\pi} g(\theta) \log(y - e^{i\theta}) \mathrm{d}\theta.$$

Moreover, we can derive the pressure P and, using $m = \frac{\partial f}{\partial h}$, the magnetisation m,

$$P = \frac{1}{\beta} \int_0^{\pi} g(\theta) \log\left(y^2 - 2y\cos\theta + 1\right) \mathrm{d}\theta,$$
$$m = 1 - 4y \int_0^{\pi} g(\theta) \frac{y - \cos\theta}{y^2 - 2y\cos\theta + 1} \mathrm{d}\theta.$$

2.4.4 A small glimple into generalisations and further areas of interest

In the decades after Lee and Yang published these ideas, their approach has been applied to a wide range of systems and many versions of their circle theorem have been found. Considering that these versions span a wide range of situations, they may not always make a statement about precisely the unit circle, so that we speak more generally of Lee-Yang theorems. We list a few examples to illustrate the diversity of further research and the significance of the foundational work by Lee and Yang. A comprehensive overview can be found in [BDL05] and [FR12].

The investigation of modified Ising model is an obvious area of interest in this context. Systems for which a Lee-Yang theorem has been proven to hold include dilute Ising ferromagnets, where the possible spin values are in $\{0, \pm 1\}$ [Suz68], and models with arbitrary even or odd continuous spin distributions [New74]. The latter is interesting not only for the sake of generality, but also because some Euclidean quantum fields can be approximated by Ising-like models with continuous spin distributions [GRS75]. For some spin systems where the spins have more than one component, a Lee-Yang theorem has been proven, such as for the plane rotator [Dun79]. An interesting approach is the one by Asano [Asa70a; Asa70b] that led to a Lee-Yang theorem for the Heisenberg model with one dominant coupling direction, see also [SF71]. The Asano method was generalised by Ruelle to other regions than the unit circle [Rue71; Rue73]. For Ising models with interactions between more than two spins, the circle theorem holds up to a certain temperature, but fails already for 4-spin interactions at high temperatures [Mon91].

The applications extend beyond equilibrium phase transitions. There have been approaches that explore the extent to which the Lee-Yang method can be applied to non-equilibrium phase transitions [DDH02].

2.4.5 The Lieb-Sokal version of a Lee-Yang theorem

One of the most generally applicable extensions of the Lee-Yang theorem was found by Lieb and Sokal [LS81]. They could show that for a model of type 2.8(b) with arbitrary spin distributions ν_i and ferromagnetic spin-spin interactions $J_{ij} \geq 0$, the complexified partition function (2.13) is non-vanishing for nonzero magnetic field whenever this is the case for the model without interactions. The theorem can be extended to encompass two-component ferromagnets with arbitrary spin distributions. The subsequent section is entirely devoted to proving this version of a Lee-Yang theorem.

3 Proof of a Lee-Yang theorem for one- and two-component ferromagnets

3.1 Introduction to the theorem and the proof

3.1.1 Remarks on notation

The purpose of this section is to prove the Lee-Yang theorem stated by Lieb and Sokal [LS81]. As we are interested in investigating the occurence of a phase transition when varying the external field h at constant inverse temperature β, we investigate the distribution of the zeros of the partition functions $\mathcal{Z}^G_{\Lambda,\beta,h}$ in the complex h-plane. We thus drop the indices Λ, β of the partition function $\mathcal{Z}^G_{\Lambda,\beta,h}$ and explicitly write out only the dependence of the partition function on the magnetic field $h = (h_1, \cdots, h_n)$, where h_i is the field at site i. The dependence on Λ and β is of course still valid. Moreover, we drop the upper index G for better readability.

3.1.2 Proof idea and goal

From Section 2.3 we recall that the partition function of an Ising-like model of the form 2.8(b) with n spins whose distributions are characterised by probability measures $\{\nu_i\}_{i=1}^n$ is given by

$$\mathcal{Z}(h_1, \cdots, h_n) = \int \exp\left(\sum_{i,j=1}^n J_{ij}\phi_i\phi_j + \sum_{i=1}^n h_i\phi_i\right)\prod_{i=1}^n \mathrm{d}\nu_i(\phi_i), \tag{3.1}$$

where J_{ij} is the interaction coefficient such that $J_{ij}\phi_i\phi_j$ equals the contribution of the interaction of the spins i and j to the Hamiltonian, and h_i denotes the external magnetic field at site i. The prefactor β in the argument of the exponential was absorbed by a rescaling of J_{ij} and h_i. The model is ferromagnetic if the alignment of two interacting spins does not increase the total energy of the system. This is the case whenever $J_{ij} \geq 0$ for all i, j.

Lieb and Sokal were able to prove that a Lee-Yang theorem holds for such models whenever it holds for the 'free' partition function

$$\mathcal{Z}_0(h_1, \cdots, h_n) = \int \exp\left(\sum_{i=1}^n h_i\phi_i\right)\prod_{i=1}^n \mathrm{d}\nu_i(\phi_i), \tag{3.2}$$

where the interaction coefficients J_{ij} are set to zero. Moreover, Lieb and Sokal showed that for even measures ν_i, the condition that all of the measures $\{\nu_i\}_{i=1}^n$ satisfiy

$$\int e^{h\phi}\mathrm{d}\nu_i(\phi) \neq 0 \quad \text{for Re } h \neq 0, \tag{3.3}$$

is a sufficient condition for the partition function (3.1) to be nonvanishing whenever Re $h \neq 0$. Before we present a proof of this theorem, we briefly outline the key steps.

The main idea of their proof is to obtain the partition function (3.1) from the free partition function (3.2) through the identity

$$\mathcal{Z}(h_1, \cdots, h_n) = \exp\left(\sum_{i,j=1}^{n} J_{ij} \frac{\partial}{\partial h_i}\frac{\partial}{\partial h_j}\right) \mathcal{Z}_0(h_1, \cdots, h_n),$$

and then prove that the differential operator

$$\exp\left(\sum_{i,j=1}^{n} J_{ij} \frac{\partial}{\partial h_i}\frac{\partial}{\partial h_j}\right) \tag{3.4}$$

preserves the nonvanishing of $\mathcal{Z}_0(h_1, \cdots, h_n)$ whenever Re $h_i > 0$ for all i. If the model is symmetric under a global spin flip (and a simultaneous reversal of the magnetic field),

$$\phi_i \to -\phi_i \quad \text{and} \quad h_i \to -h_i \quad \text{for all } i, \tag{3.5}$$

then it is immediately implied that the partition function has no zeros with negative real part and that there is no phase transition at nonzero real magnetic field. The symmetry (3.5) manifests itself in the probability measures $\{\nu_i\}_{i=1}^n$. The model has a symmetry under the transformation (3.5) if and only if all measures ν_i are even.

Before delving into the details of the proof, we present the Lee-Yang theorem in the form in which it is proven below. We say that a measure ν has the *Lee-Yang property* if the free partition function that it defines as in (3.2) is nonvanishing whenever Re $h > 0$. The *falloff* is a technical term that entails a statement about the exponential order and type of the resulting partition function. It is not directly relevant to the physical interpretation of the theorem and will therefore not be explained further at this point.

Theorem. *Let $\{\nu_i\}_{i=1}^n$ be measures on $\mathbb{R}^1$, each having the Lee-Yang property with falloff β. Let J be a symmetric $n \times n$ matrix with nonnegative entries and $\|J\| < \beta$. Then the measure μ on $\mathbb{R}^n$ given by*

$$\mathrm{d}\mu(\phi) = \exp\left(\sum_{i,j=1}^{n} J_{ij}\phi_i\phi_j\right) \prod_{i=1}^{n} \mathrm{d}\nu_i(\phi_i)$$

has the Lee-Yang property with falloff γ for every $\gamma < \beta - \|J\|$ (and $\gamma = \infty$ if $\beta = \infty$). In particular, we can let each measure ν_i be an even measure satisfying

$$\int e^{h\phi}\mathrm{d}\nu_i(\phi) \neq 0 \text{ for Re } h \neq 0.$$

3.1.3 Overview of the proof

The Lee-Yang theorem that we prove is a consequence of a more general statement about certain entire functions. Following Lieb and Sokal's proof, we establish this statement first and then apply it to the situation described above. Roughly speaking, the statement looks as follows.

Denote $D^n := \{z \in \mathbb{C}^n \colon \mathrm{Re}\ z_i > 0$ *for all* $i\}$. *Let* f, g *be in a suitable class of functions on* $\mathbb{C}^n$ *and both nonvanishing on* D^n. *Then* $f(\partial_z)g(z)$ *is well-defined and*

$$f(\partial_z)g(z) \neq 0 \quad \text{for all } z \in D^n. \tag{3.6}$$

The precise formulation of this statement can be found in Proposition 3.10.

Such results were already known for complex polynomials when Lieb and Sokal published their proof of a Lee-Yang theorem for general ferromagnetic models (see for example pp. 82-84 in [Mar66] which is cited by Lieb and Sokal, after the proof of their Proposition 2.1). Their approach provides a generalisation of this result to a class of functions that contains the partition functions of the above-described models.

To obtain this generalisation, we start by extending the result for polynomials to polynomials in n complex variables, this is the content of our Proposition 3.1. In Section 3.3, we introduce the suitable space of functions, it depends on a parameter $a \geq 0$ and is denoted by $\mathscr{A}^n_{a+}$. This space contains entire functions growing more slowly than e^{bz^2} for all $b > a$. Corollary 3.8 states that for $f, g \in \mathscr{A}^n_{a+}, \mathscr{A}^n_{b+}$ respectively, the formal expression (3.6) gives a well-defined function if $ab < \frac{1}{4}$, and this function is in $\mathscr{A}^n_{c+}$ for a suitable c.

One of the central properties of the space $\mathscr{A}^n_{a+}$ is that the polynomials are dense in it. This property holds great significance as it enables us to derive the desired statement about functions in $\mathscr{A}^n_{a+}$ from the case of polynomials. In Proposition 3.10 we show that the property (3.6) is preserved by an approximation by polynomials. The spaces $\overline{\mathscr{P}^n_{a+}(D^n)}$ will be introduced as the subspaces of functions in $\mathscr{A}^n_{a+}$ that are approximable by polynomials nonvanishing on D^n, the product of right half-planes. Clearly, the property (3.6) carries over from polynomials to functions in $\overline{\mathscr{P}^n_{a+}(D^n)}$. In particular, Proposition 3.11 shows that for a symmetric matrix J, the function of interest for a Lee-Yang theorem,

$$\exp\left(\sum_{i,j=1}^{n} J_{ij}\phi_i\phi_j\right) = \lim_{k\to\infty}\prod_{i,j=1}^{n}\left(1 + \frac{J_{ij}\phi_i\phi_j}{k}\right)^k,$$

is in $\overline{\mathscr{P}^n_{a+}(D^n)}$ for any $a > \|J\|$ if and only if $J_{ij} \geq 0$ for all i, j. This is precisely the situation that occurs in the partition function of a ferromagnet.

To derive the Lee-Yang theorem, it only remains to show that the free partition function can also be described in terms of the spaces $\mathscr{A}^n_{a+}$.

To this end, we understand the spin measures ν_i as distributions in the sense that they act on functions in the following way,

$$f \mapsto \int_{\mathbb{R}} f(\phi)\mathrm{d}\nu_i(\phi).$$

Clearly, in the context of our application, this operation must be well-defined for functions

$$f(\phi) = e^{J\phi^2 + h\phi}, \quad J \geq 0, \tag{3.7}$$

or more generally, for functions of the form $f(\phi) = e^{J\phi^2+h\phi}g(\phi)$ with $J \geq 0$, where $g \in \mathscr{S}(\mathbb{R})$ is a Schwartz function. It is thus natural to consider distributions of the form

$$T = e^{-\beta x^2} T_\beta \tag{3.8}$$

for some $\beta > 0$ and a tempered distribution T_β, where the action of (3.8) on a function f is understood as the action of T_β on the product of f and a function ζ_β, where $\zeta_\beta(x) = e^{-\beta x^2}$,

$$T(f) = T_\beta(\zeta_\beta f).$$

Specifically, if we choose $\beta > J$, this gives a well-defined distribution for functions of the form (3.7) because

$$\left| \int_{\mathbb{R}} f(\phi) e^{-\beta\phi^2} \mathrm{d}\nu_i(\phi) \right| \leq \sup_\phi \left(e^{-(\beta-J)\phi^2+h\phi} \right) \nu_i(\mathbb{R}) < \infty.$$

A formal definition of the distribution spaces $\mathscr{T}_\beta^n$ is given in Section 3.3.3.

The Laplace transform will be used to turn the distributions into functions of a complex variable. In Lemma 3.15, it is shown that for $T \in \mathscr{T}_\beta^n$, the Laplace transform $\hat{T}$ is in $\mathscr{A}^n_{1/4\beta+}$, and if, in addition, $\hat{T}$ is approximable by polynomials that are nonvanishing on D^n, Proposition 3.16 implies that the property (3.6) holds for the application of the differential operator $\exp\left(\sum_{i,j=1}^n J_{ij} \frac{\partial}{\partial h_i} \frac{\partial}{\partial h_j} \right)$ to $\hat{T}$.

Once these mathematical statements are proven, it is easy to deduce the Lee-Yang theorem. This will be done in a general form in Theorem 3.18. Corollary 3.19 is then precisely the Lee-Yang theorem for the one-component ferromagnet that was presented above. It is only a minor effort to show that the assumption (3.3) is sufficient to guarantee that the free partition function is approximable by polynomials nonvanishing in D^n.

The methods can be extended to prove a Lee-Yang theorem for a general two-component ferromagnet. The domain where the Laplace transform of a measure does not vanish is different in that case. It is determined in Proposition 3.22. A change of variables transforms this domain into a product of two regions D^n, which leads us back to the situation studied in Sections 3.2-3.4. The previously derived theorems carry over immediately and it follows that a Lee-Yang theorem holds in terms of the new variables if all interaction coefficients are nonnegative. It is easy to transform the variables back to those of the two-component ferromagnet partition function and to derive the conditions on the interaction coefficients under which a Lee-Yang theorem holds by inserting the nonnegativity of the other interaction coefficients into the transformation. The resulting Lee-Yang theorem is Corollary 3.25.

3.2 The case of polynomials

We now proceed to presenting the proofs of the outlined ideas. In the preceding paragraph, we explained that the Lee-Yang theorem can be proven by demonstrating that the differential operator (3.4) applied to a suitable function preserves its nonvanishing property in D^n. To establish this, we will approximate both the differential operator and the function by polynomials, therefore such a statement needs to be shown to hold for polynomials first. Proving this result, which is our Proposition 3.1, is the goal of this section.

Proposition 3.1. *Let P, Q be polynomials in n complex variables and define for $v, w, z \in \mathbb{C}^n$*

$$R(v, w) = P(v)Q(w)$$

and

$$S(z) = P(\partial/\partial z)Q(z).$$

If $R(v, w) \neq 0$ whenever $\operatorname{Re} v > 0$ and $\operatorname{Re} w > 0$, then it holds that $S(z) \neq 0$ whenever $\operatorname{Re} z > 0$ or else $S(z)$ is identically zero.

The proof strategy relies on the following identity that allows us to convert the expression for R into S,

$$S(w) = \prod_{i=1}^{n} \exp\left(\frac{\partial}{\partial v_i}\frac{\partial}{\partial w_i}\right) R(v, w)\bigg|_{v=0}.$$

We will write the exponential in the product as a function and approximate it by products of degree-one polynomials. The degree-one polynomials will successively be converted into differential operators by replacing the indeterminate with derivatives. The next lemma shows that this replacement preserves the nonvanishing of the expression in $\{z \in \mathbb{C} \colon \operatorname{Re} z \geq 0\}$. We will finish the proof by arguing that the nonvanishing property carries over when taking the limit of the product of polynomials to the exponential, or else the resulting expression vanishes identically. Before carrying out the proof, we establish the necessary statement for degree-one polynomials.

Lemma 3.2. *Let Q_0 and Q_1 be polynomials in a single complex variable, and assume that $R(v, w) = Q_0(w) + vQ_1(w) \neq 0$ whenever $\operatorname{Re} v \geq 0$ and $\operatorname{Re} w \geq 0$. Then $S(z) = Q_0(z) + Q_1'(z) \neq 0$ whenever $\operatorname{Re} z \geq 0$.*

Proof. Five claims will be made and proven first. With the claims, the proof of the lemma is simple.

Claim 1. $Q_0(z) \neq 0$ whenever $\operatorname{Re} z \geq 0$.

Proof of Claim 1. Per assumption, $R(0, w) \neq 0$ must hold, so $Q_0(z) \neq 0$ whenever $\operatorname{Re} z \geq 0$. If $vQ_1 \equiv 0$, we are done, so we may assume $Q_1 \not\equiv 0$ and $v \neq 0$.

Claim 2. $Q_1(z) \neq 0$ whenever $\operatorname{Re} z > 0$.

Proof of Claim 2. Define $\tilde{R}(x, w) := xQ_0(w) + Q_1(w)$ and set $x = 1/v$ (obviously, $\tilde{R}$ can be extended to an analytic function in x on all of $\mathbb{C}$). Then $\tilde{R}(x, w) = \frac{1}{v}R(v, w)$ and the zeros (x_0, w_0) of $\tilde{R}$ and the zeros (v_0, w_0') of R are related by $(x_0, w_0) = (1/v_0, w_0')$. Note that $\operatorname{Re} x = \operatorname{Re} v / |v|^2$ which implies that $\operatorname{Re} x$ and $\operatorname{Re} v$ have the same sign. Hence $\tilde{R}$ has no zeros whenever $\operatorname{Re} x > 0, \operatorname{Re} w \geq 0$ per assumption on R.

Suppose the claim to be false, that is, there exists some z_0 with Re $z_0 > 0$ and $Q_1(z_0) = 0$. Then it holds that $\tilde{R}(0, z_0) = 0$. Define the polynomials $\tilde{R}_k(w) = \tilde{R}(1/k, w)$. These form a sequence $(\tilde{R}_k)_{k\in\mathbb{N}}$ of entire functions converging uniformly to $\tilde{R}(0, \cdot)$ on compact subsets $K \subset \mathbb{C}$. It is easy to see the uniform convergence on compacts. Considering that polynomials are bounded on compact subsets $K \subset \mathbb{C}$, we find that for all $\varepsilon > 0$ there exists some $k_0 \in \mathbb{N}$ such that $k_0 > \varepsilon^{-1} \max_{z\in K} |Q_0(z)|$. For all $k \geq k_0$, this yields

$$\left|\tilde{R}_k(z) - \tilde{R}(0, z)\right| = \left|\frac{1}{k} Q_0(z)\right| < \varepsilon$$

for all $z \in K$ which proves the uniform convergence on compacts.

Per assumption, z_0 is a zero of $\lim_{k\to\infty} \tilde{R}_k = \tilde{R}(0, \cdot)$, therefore by the Hurwitz theorem (Theorem 7.2 in [FB06]), every sufficiently small neighbourhood of $(0, z_0)$ contains a zero z_k of $\tilde{R}_k$ for k sufficiently large. In particular, we can make the neighbourhood so small that Re $z_k > 0$ because Re $z_0 > 0$. This, however, implies that (k, z_k) is a zero of R in the right half-plane D which contradicts the assumption on R. Therefore, $Q_1(z_0) \neq 0$ whenever Re $z_0 > 0$.

Claim 3. If $Q_1(z_0) = 0$ and Re $z_0 = 0$, then $\frac{Q_1'(z_0)}{Q_0(z_0)}$ is real and nonnegative.

Proof of Claim 3. Let Re $z_0 = 0$. If $Q_1(z_0) = 0$, then $\tilde{R}(0, z_0) = 0$. The implicit function theorem can be applied to $\tilde{R}$ because $\partial_x \tilde{R}(x, w)|_{(0,z_0)} = Q_0(z_0) \neq 0$ according to Claim 1. It implies that there is an open neighbourhood U of z_0 and an analytic function $g\colon U \to \mathbb{R}$ where $x = g(w)$, $\tilde{R}(g(w), w) = 0$ and $\partial_w g(w) = -\left(\partial_x \tilde{R}\right)^{-1} \partial_w \tilde{R}|_{(g(w),w)}$.

Calculating the derivative explicitly yields

$$\partial_w g(w) = -\left(\partial_x \tilde{R}\right)^{-1} \partial_w \tilde{R}|_{(g(w),w)} = -\frac{g(w)Q_0'(w) + Q_1'(w)}{Q_0(w)}.$$

Recall that $g(z_0) = 0$ and perform a Taylor expansion up to first order to write

$$g(z_0 + \delta w) = g(z_0) + \partial_w g(z_0)\delta w + r = -\frac{Q_1'(z_0)}{Q_0(z_0)}\delta w + r =: \delta x,$$

where $r = \mathcal{O}(\delta w^2)$. Suppose $\frac{Q_1'(z_0)}{Q_0(z_0)}$ is not both real and nonnegative. Then it is possible to choose δw with $z_0 + \delta w \in U$ and Re $\delta w > 0$ such that $-\frac{Q_1'(z_0)}{Q_0(z_0)}\delta w$ has positive real part. As $|\delta w|$ can be arbitrarily small, the second-order term r is negligible for determining the real part of δx, so that Re $\delta x > 0$ follows.

In U it must hold that $\tilde{R}(g(w), w) = 0$, which implies $\tilde{R}(g(z_0 + \delta w), z_0 + \delta w) = 0$. This yields a zero of $\tilde{R}$ at $(\delta x, z_0 + \delta w)$ where both Re δx and Re δw are positive, and therefore a zero of R in the right half-plane which contradicts the assumption on R. So $\frac{Q_1'(z_0)}{Q_0(z_0)}$ is real and nonnegative.

Claim 4. Re $\frac{Q_0(z)}{Q_1(z)} > 0$ whenever Re $z \geq 0$ and $Q_1(z) \neq 0$.

Proof of Claim 4. Suppose that the claim does not hold true. Then there exists a zero of $R(v, w)$ with Re $v \geq 0$ and Re $w \geq 0$ by choosing $v = -\frac{Q_0(z)}{Q_1(z)}$, contrary to the assumption on R.

Claim 5. Let Re $z \geq 0$ and $Q_1(z) \neq 0$. Then Re $\frac{Q_1'(z)}{Q_1(z)} \geq 0$.

Proof of Claim 5. Factor Q_1 into

$$Q_1(z) = b \prod_{j=1}^{\deg Q_1} (z - \beta_j),$$

where β_j are the zeros of Q_1. Claim 2 shows that Re $\beta_j \leq 0$ for all j. Then, as $Q_1(z) \neq 0$ per assumption, the quotient

$$\frac{Q_1'(z)}{Q_1(z)} = \sum_{j=1}^{\deg Q} \frac{1}{z - \beta_j}$$

is well-defined. Recall that for $x \neq 0$, the real part of x has the same sign as $\mathrm{Re}(x^{-1})$, thus every term of the sum has nonnegative real part and therefore Re $\frac{Q_1'(z)}{Q_1(z)} \geq 0$.

Proof of the lemma. Now the proof of the lemma is a simple conclusion of the previous claims.

First assume $Q_1(z) \neq 0$ and Re $z \geq 0$. Then

$$Q_0(z) + Q_1'(z) = Q_1(z) \left(\frac{Q_0(z)}{Q_1(z)} + \frac{Q_1'(z)}{Q_1(z)} \right) \neq 0,$$

because Re $\frac{Q_0(z)}{Q_1(z)} > 0$ (Claim 4) and Re $\frac{Q_1'(z)}{Q_1(z)} \geq 0$ (Claim 5).

Now assume $Q_1(z) = 0$. Claim 2 implies that Re $z = 0$. As $Q_0(z) \neq 0$ whenever Re $z \geq 0$ from Claim 1, it must hold that

$$Q_0(z) + Q_1'(z) = Q_0(z) \left(1 + \frac{Q_1'(z)}{Q_0(z)} \right) \neq 0,$$

because Re $\frac{Q_1'(z)}{Q_0(z)} \geq 0$ according to Claim 3. □

With Lemma 3.2, we are now equipped to prove Proposition 3.1 with the above-outlined proof strategy. For $v \in \mathbb{C}^n$, we introduce the notation Re $v > 0$ to abbreviate Re $v_i > 0$ for $i = 1, \cdots, n$.

Proof of Proposition 3.1. To prove that the identity

$$S(w) = \prod_{i=1}^{n} \exp \left(\frac{\partial}{\partial v_i} \frac{\partial}{\partial w_i} \right) R(v, w) \Bigg|_{v=0}$$

holds, we write the polynomial R as a sum of monomials,

$$R(v, w) = \sum_{l,k=1}^{n} \sum_{m_1=0}^{\deg_{v_l} R} \sum_{m_2=0}^{\deg_{w_k} R} \alpha_{m_1 m_2}^{(lk)} v_l^{m_1} w_k^{m_2},$$

with coefficients $\alpha_{m_1 m_2}^{(lk)} \in \mathbb{C}$.

The differential operator $\exp\left(\frac{\partial}{\partial v_i}\frac{\partial}{\partial w_i}\right)$ is defined by a power series,

$$\exp\left(\frac{\partial}{\partial v_i}\frac{\partial}{\partial w_i}\right) = \sum_{n\in\mathbb{N}_0} \frac{\left(\frac{\partial}{\partial v_i}\frac{\partial}{\partial w_i}\right)^n}{n!}$$

It is easy to see that only finitely many terms of the power series yield a nonzero result when applied to the polynomial $R(v, w)$.

We first evaluate $S(w)$ so that we can later compare it to the result of the application of the differential operator to R. The linearity of differentiation allows us to reduce the proof of the identity to the case that R is a product of monomials in v_k, w_l. Let therefore $R(v, w) = v_k^{m_1} w_l^{m_2}$. If $m_1 \le m_2$, then

$$S(w) = \frac{\partial^{m_1}}{\partial {w_k}^{m_1}} w_l^{m_2} = \delta_{kl}\; m_2(m_2-1)\cdots(m_2-m_1+1) w_l^{(m_2-m_1)},$$

and otherwise $S(w) = 0$. On the other hand,

$$\sum_{n\in\mathbb{N}_0} \frac{\left(\frac{\partial}{\partial v_i}\frac{\partial}{\partial w_i}\right)^n}{n!} v_k^{m_1} w_l^{m_2}$$

$$= \delta_{ik}\delta_{il} \sum_{n=0}^{\min\{m_1,m_2\}} \frac{1}{n!} m_1(m_1-1)\cdots(m_1-n+1) v_k^{m_1-n} m_2(m_2-1)\cdots(m_2-n+1) w_l^{m_2-n},$$

so that after an evaluation at $v_k = 0$, only the term with $n = m_1$ remains. If $m_2 < m_1$, then no such term exists and the expression vanishes. If, however, $m_1 \le m_2$, then the remaining term is given by

$$\delta_{ik}\delta_{il}\; m_2(m_2-1)\cdots(m_2-m_1+1) w_l^{m_2-m_1}.$$

In both cases it is equal to the expression for $S(w)$, so that the identity is proven.

Now we return to the proof of the proposition. Suppose $R(v, w) \ne 0$ whenever $\operatorname{Re} v > 0, \operatorname{Re} w > 0$. Our goal is to show that the application of this differential operator preserves the nonvanishing of R in D^n. As in Lemma 3.2, we consider the product of two functions $\prod_{i=1}^{n} \exp(s_i u_i) R(v, w)$ and then replace the indeterminates in the argument of $\prod_{i=1}^{n} \exp(s_i u_i)$ by partial derivatives to obtain $S(w)$. For clarity we set $F(s, u) = \prod_{i=1}^{n} \exp(s_i u_i)$ and write

$$F(s, u)R(v, w) = \prod_{i=1}^{n} \exp(s_i u_i) R(v, w).$$

We want to prove that after replacing s_i, u_i by $\partial_{v_i}, \partial_{w_i}$, the result is nonvanishing on $\{0\}^n \times D^n$ by a repeated application of Lemma 3.2. The lemma applies to polynomials, therefore we approximate the exponential by polynomials using $\exp(x) = \lim_{m\to\infty} (1 + \frac{x}{m})^m$. Note that the approximation is uniform on compacts. Explicitly, we write F as the limit of the following polynomials,

$$F(s,u) = \lim_{m\to\infty} \left(\prod_{i=1}^{n} \left[1 + \frac{s_i u_i}{m} \right] \right)^m . \tag{3.9}$$

The expression can be decomposed into a product of degree-one polynomials. In the resulting form, we can repeatedly apply Lemma 3.2 to successively convert the variables into derivatives. We define the degree-one polynomials P_m in the complex variables s_i and u_i by $P_m(s_i,u_i) = 1 + m^{-1}s_iu_i$. Then F is the limit of polynomials F_m, where

$$F_m(s,u) := \left(\prod_{i=1}^{n} \left[1 + m^{-1} s_i u_i \right] \right)^m = \prod_{i=1}^{n} \left(P_m(s_i,u_i) \right)^m .$$

In order to apply Lemma 3.2, the expression must first be modified such that $F_m(s,u)R(v,w) \neq 0$ whenever $\mathrm{Re}\ s, \mathrm{Re}\ u, \mathrm{Re}\ v, \mathrm{Re}\ w \geq 0$. Therefore we shift all variables by $\varepsilon > 0$ to obtain

$$\prod_{i=1}^{n} \left(P_m(s_i+\varepsilon, u_i-\varepsilon) \right)^m R(v+\varepsilon, w+\varepsilon). \tag{3.10}$$

Denote $R^{(\varepsilon)}(v,w) := R(v+\varepsilon, w+\varepsilon)$. Then $R^{(\varepsilon)}$ is nonvanishing whenever $\mathrm{Re}\ v \geq 0, \mathrm{Re}\ w \geq 0$ per assumption on R.

Moreover, we claim that there are no zeros of $P_m(s,u) := (1+m^{-1}(s)(u))$ with $\mathrm{Re}\ s > 0, \mathrm{Re}\ u > 0$. To see this, suppose the claim is false and there is a zero (s_0,u_0) of the polynomial P_m where we write $s_0 = a+ib, u_0 = c+id$ with $a,c > 0$. Then

$$0 = P_m(s_0,u_0) = (1 + m^{-1}(a+ib)(c+id)) = 1 + m^{-1}\left[(ac-bd) + i\,(ad+bc) \right].$$

Both the real and imaginary part of this expression must vanish separately. Solving the resulting equation for the imaginary part yields $d = -\frac{bc}{a}$. After inserting this d into the equation for the real part, we see that such a zero must satisfy

$$ma = -c(b^2+a^2).$$

However, this leads to a contradiction because $m \in \mathbb{N}$ and we assumed that a and c are positive. Hence, $P_m(s,u) := (1+m^{-1}(s)(u)) \neq 0$ whenever $\mathrm{Re}\ s > 0, \mathrm{Re}\ u > 0$ and it follows that $P_m^{(\varepsilon)}(s,u) := P_m(s+\varepsilon, u+\varepsilon)$ is nonvanishing whenever $\mathrm{Re}\ s \geq 0, \mathrm{Re}\ u \geq 0$.

The next step is to convert the variables into derivatives and apply Lemma 3.2. We therefore fix $(v_j)_{j\neq i}, (w_j)_{j=1,\cdots,n}, u_i$ with nonnegative real part and write

$$\begin{aligned} P_m^{(\varepsilon)}(s_i,u_i)R^{(\varepsilon)}(v,w) &= 1 + m^{-1}(s_i+\varepsilon)(u_i+\varepsilon))R^{(\varepsilon)}(v,w) \\ &= \underbrace{\left(1 + m^{-1}(\varepsilon u_i + \varepsilon^2)R^{(\varepsilon)}(v,w)\right)}_{=:Q_0(v_i)} + s_i \underbrace{\left(m^{-1}(u_i+\varepsilon)R^{(\varepsilon)}(v,w)\right)}_{=:Q_1(v_i)}. \end{aligned}$$

It is obvious that $Q_0(v_i) + s_i Q_1(v_i) \neq 0$ whenever Re v_i, Re $s_i \geq 0$ because this holds for both $P^{(\varepsilon)}$ and $R^{(\varepsilon)}$. Now Lemma 3.2 can be applied to show that

$$P_m^{(\varepsilon)}(\partial_{v_i}, u_i) R^{(\varepsilon)}(v, w) \neq 0$$

whenever Re $v_i \geq 0$ for the values of $(v_j)_{j \neq i}, (w_j)_{j=1,\cdots,n}, u_i$ that were fixed above. These were arbitrary, therefore

$$P_m^{(\varepsilon)}(\partial_{v_i}, u_i) R^{(\varepsilon)}(v, w) \neq 0$$

whenever Re v, Re w, Re $u_i \geq 0$.

In the resulting expression, we fix $(v_j)_{j=1,\cdots,n}, (w_j)_{j \neq i}$ with nonnegative real part and rearrange

$$P_m^{(\varepsilon)}(\partial_{v_i}, u_i) R^{(\varepsilon)}(v, w) = \underbrace{\Big(1 + m^{-1}(\varepsilon \partial_{v_i} + \varepsilon^2) R^{(\varepsilon)}(v, w)\Big)}_{=: q_0(w_i)} + u_i \underbrace{\Big(m^{-1}(\partial_{v_i} + \varepsilon) R^{(\varepsilon)}(v, w)\Big)}_{=: q_1(w_i)}.$$

As before, it holds that $q_0(v_i) + u_i q_1(v_i) \neq 0$ whenever Re v_i, Re $u_i \geq 0$. Lemma 3.2 implies that

$$P_m^{(\varepsilon)}(\partial_{v_i}, \partial_{w_i}) R^{(\varepsilon)}(v, w) \neq 0$$

whenever Re $w_i \geq 0$ and, by the same reasoning as above, the expression is nonvanishing whenever Re v, Re $w \geq 0$.

So far, (3.10) has been transformed to

$$\prod_{\substack{j=1 \\ i \neq j}}^{n} (P_m(s_j + \varepsilon, u_j + \varepsilon))^m (P_m(s_i + \varepsilon, u_i + \varepsilon))^{m-1} P_m^{(\varepsilon)}(\partial_{v_i}, \partial_{w_i}) R(v + \varepsilon, w + \varepsilon),$$

which is still nonzero whenever Re v, Re w, Re s, Re $u \geq 0$. The previous procedure can be repeated to convert the remaining factors that contain polynomials P_m successively into derivatives. After each step, it holds that the expression is nonzero whenever Re v, Re w, Re s, Re $u \geq 0$. In particular,

$$\prod_{\substack{j=1 \\ i \neq j}}^{n} P_m^{(\varepsilon)}(\partial_{v_i}, \partial_{w_i})^m R(v + \varepsilon, w + \varepsilon)\Big|_{v=0} \neq 0.$$

Therefore, (3.10) can be converted into

$$\left(\prod_{i=1}^{n} \left[1 + m^{-1} \left(\frac{\partial}{\partial v_i} + \varepsilon \right) \left(\frac{\partial}{\partial w_i} + \varepsilon \right) \right] \right)^m R^{(\varepsilon)}(v, w), \tag{3.11}$$

which is nonzero whenever Re v, Re $w \geq 0$. Finally, as R is a polynomial, the application of products of differential operators of the form $a + c\partial_{v_i}, b + d\partial_{w_i}$ (where a, b, c, d are arbitrary complex numbers) on R yields a polynomial. If a sequence of polynomials converges to some limit, it converges uniformly on compacts. Therefore, the Hurwitz theorem applies and shows that either

$$S^{(\varepsilon)}(w) := \lim_{m\to\infty} \left(\left\{ \prod_{i=1}^{n} \left[1 + m^{-1} \left(\frac{\partial}{\partial v_i} + \varepsilon \right) \left(\frac{\partial}{\partial w_i} + \varepsilon \right) \right] \right\}^{m} \right) R(v+\varepsilon, w+\varepsilon) \Bigg|_{v=0} \neq 0$$

whenever Re $w > 0$, or else the expression vanishes identically. We can use the Hurwitz theorem again to show that $S^{(\varepsilon)}(w)$ can vanish identically for some $\varepsilon > 0$ only if it vanishes for all $\varepsilon > 0$. The convergence of (3.11) in the limit $\varepsilon \to 0$ is also uniform on compacts since the resulting polynomials will converge to the corresponding expression in non-shifted variables, and this is still true when we take ε to be a complex variable. We can therefore exchange the limits $m \to \infty$ and $\varepsilon \to 0$ and see that $S^{(\varepsilon)}(w) \to S(w)$. Define $S_m(w, \varepsilon)$ by (3.11). We know that $S_m(w, \varepsilon) \neq 0$ whenever Re $w \geq 0$ and Re $\varepsilon > 0$ (the imaginary part of ε can be thought of as a shift in the imaginary part of v and w, and since we can choose the imaginary part of v and w in the above arguments arbitrarily, the nonvanishing holds true for any such purely imaginary shift). If now (w, ε_0) is a zero of $\lim\limits_{m\to\infty} S_m$ for some ε_0 with Re $\varepsilon_0 > 0$ and Re $w > 0$, then by the Hurwitz theorem it follows that $\lim\limits_{m\to\infty} S_m \equiv 0$. In particular, $S^{(\varepsilon)}(w)$ vanishes identically for some $\varepsilon > 0$ only if it vanishes identically for all $\varepsilon > 0$.

If $S^{(\varepsilon)}$ is already identically zero for every $\varepsilon > 0$, then this will also be the case in the limit $\varepsilon \to 0$ and the proposition is proven. Else we can apply the Hurwitz theorem again to show that either

$$S(w) = \lim_{\varepsilon\to 0} S^{(\varepsilon)}(w) \neq 0$$

whenever Re $w > 0$, or else $S \equiv 0$. □

3.3 Generalisation to a class of entire functions

3.3.1 The space $\mathscr{A}^n_{a+}$ and its properties

In the first part of this section, the space $\mathscr{A}^n_{a+}$ is introduced and some properties of its functions are discussed. It will be shown that the polynomials are dense in it. We will use this property in Section 3.3.2 to derive Proposition 3.10, which is an analogue of Proposition 3.1, by employing approximations with polynomials.

Definition 3.3. Let f be an entire function and $b > 0$. Define

$$\|f\|_b = \sup_{z\in\mathbb{C}^n} \left\{ \exp\left(-b \sum_{i=1}^{n} |z_i|^2 \right) |f(z)| \right\}.$$

For $a \geq 0$, let $\mathscr{A}^n_{a+}$ be the space of entire functions such that $\|f\|_b < \infty$ for all $b > a$. Equipped with a countable family of norms $(\|\cdot\|_b)_{b\in\mathbb{Q}\cap\{b>a\}}$, this is a Fréchet space (for a reminder of the definition and some facts about Fréchet spaces, see Section A in the appendix).

Functions in $\mathscr{A}^n_{a+}$ are entire functions, hence they can be represented by the power series

$$f(z) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^n} \alpha_{\boldsymbol{k}} z^{\boldsymbol{k}}.$$

The Cauchy integral fomula yields a useful estimate for the power series coefficients $\alpha_{\boldsymbol{k}}$ of a function with finite $\|\cdot\|_b$-norm.

Lemma 3.4. *Let $b > 0$ and f be an entire function such that $\|f\|_b < \infty$. For the coefficients in the power series expansion*

$$f(z) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^n} \alpha_{\boldsymbol{k}} z^{\boldsymbol{k}}$$

it holds that

$$|\alpha_{\boldsymbol{k}}| \leq \|f\|_b \prod_{i=1}^n \left(\frac{2eb}{k_i}\right)^{k_i/2}.$$

Proof. The lemma will be shown for $n = 1$ first.

It is clear that $\alpha_k = \frac{f^{(k)}(0)}{k!}$ (see for example Theorem III.2.2 in [FB06]). Using the Cauchy integral formula, write the k^{th} derivative as

$$f^{(k)}(0) = \frac{k!}{2\pi i}\oint_\gamma \frac{f(w)}{w^{k+1}}\,\mathrm{d}w = \frac{k!}{2\pi i}\int_0^1 \frac{f(\gamma(t))}{\gamma(t)^{k+1}}\dot\gamma(t)\,\mathrm{d}t.$$

Choosing the curve γ of the form $\gamma(t) = z_0 + re^{2\pi it}$ with $t \in [0,1)$ and $|z_0| < r$, it holds that $\dot\gamma(t) = 2\pi i r e^{2\pi it}$. We can choose $z_0 = 0$, then

$$\left|f^{(k)}(0)\right| \leq \frac{k!}{2\pi r^{k+1}}\Big|\int\limits_{|w|=r} f(w)\,\mathrm{d}w\Big|. \tag{3.12}$$

Clearly,

$$\sup_{|w|=r} e^{-b|w|^2}|f(w)| \leq \sup_{w\in\mathbb{C}} e^{-b|w|^2}|f(w)| = \|f\|_b\,,$$

which shows

$$\sup_{|w|=r}|f(w)| \leq e^{br^2}\|f\|_b\,. \tag{3.13}$$

Therefore, using the standard estimate for complex curve integrals (Remark II.1.5.2 in [FB06]),

$$\Big|\int\limits_\gamma f(w)\,\mathrm{d}w\Big| \leq L(\gamma)\sup_\gamma|f(w)|\,,$$

it follows that, using (3.12) and (3.13),

$$\left|f^{(k)}(0)\right| \leq k! e^{br^2} \|f\|_b \frac{1}{r^k}. \tag{3.14}$$

This bound must hold for all $r > 0$. We determine the value of r for which it is minimised. A simple calculation shows

$$\frac{\mathrm{d}}{\mathrm{d}r} \frac{e^{br^2}}{r^k} = \left(\frac{-k}{r^{k+1}} + \frac{2br}{r^k} \right) e^{br^2},$$

which vanishes if and only if $r = \sqrt{\frac{k}{2b}}$. Inserting this value of r into (3.14) yields

$$\left|f^{(k)}(0)\right| \leq k! \|f\|_b \left(\frac{2b}{k} \right)^{k/2},$$

and therefore $|\alpha_k| \leq \|f\|_b \left(\frac{2b}{k}\right)^{k/2}$.

The generalisation to higher n is proven analogously to the above proof, noting that the power series of an entire function in multiple variables can be written as

$$f(z) = \sum_{k_1, \cdots, k_n = 0}^{\infty} c_{k_1, \cdots, k_n} z_1^{k_1} \cdots z_n^{k_n} \ ,$$

where

$$c_{k_1 \cdots k_n} = \frac{1}{(2\pi i)^n} \int_{\partial\gamma_1} \cdots \int_{\partial\gamma_n} \frac{f(w_1, \cdots, w_n)}{w_1^{k_1+1} \cdots w_n^{k_n+1}} \, dw_1 \cdots dw_n.$$

This follows from repeated application of the Cauchy integral formula, see Theorem [Fre14] V.2.2. □

The next proposition summarises some properties of functions in the spaces $\mathscr{A}^n_{a+}$.

Proposition 3.5. *For all $a \geq 0$, the space $\mathscr{A}^n_{a+}$ has the following properties.*

(a) $\mathscr{A}^n_{a+}$ is closed under differentiation.

(b) A bounded sequence f_j in $\mathscr{A}^n_{a+}$ converges pointwise to $f \in \mathscr{A}^n_{a+}$ if and only if it converges to f in the topology of $\mathscr{A}^n_{a+}$.

(c) For $f \in \mathscr{A}^n_{a+}$, the partial sums of the Taylor series of f converge to f in the topology of $\mathscr{A}^n_{a+}$.

Proof. First we prove *(a)* which is a direct consequence of the Cauchy integral formula. The proof will be given for $n = 1$ only. As in the proof of Lemma 3.4, the generalisation to higher n is straightforward. Let $a \geq 0$ and $f \in \mathscr{A}^n_{a+}$. We write, using the Cauchy integral formula,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(w)}{(w-z)^{k+1}} \, \mathrm{d}w = \frac{k!}{2\pi i} \int_0^1 \frac{f(\gamma(t))}{(\gamma(t) - z)^{k+1}} \dot{\gamma}(t) \, \mathrm{d}t.$$

Let $b > a$. Then

$$\left|f^{(k)}(z)\right| e^{-b\left|z^2\right|} = \frac{k!}{2\pi}\left|\int_0^1 \frac{f(\gamma(t))e^{-b\left|z^2\right|}}{(\gamma(t)-z)^{k+1}}\dot\gamma(t)\,\mathrm{d}t\right|.$$

As $f \in \mathscr{A}^n_{a+}$, it must hold that

$$\sup_{t\in[0,1]} f(\gamma(t))e^{-b|z|^2} = \sup_{t\in[0,1]} f(\gamma(t))e^{-b|\gamma(t)|^2}e^{-b(|z|^2-\gamma(t)^2)} \leq \|f\|_b \sup_{t\in[0,1]} e^{-b(|z|^2-|\gamma(t)|^2)}.$$

Inserting this into the estimate of $\left|f^{(k)}(z)\right|$ yields

$$\left|f^{(k)}(z)\right| e^{-b\left|z^2\right|} \leq \frac{k!}{2\pi}\|f\|_b \int_0^1 \left|\frac{e^{-b(|z|^2-|\gamma(t)|^2)}}{(\gamma(t)-z)^{k+1}}\dot\gamma(t)\right| \mathrm{d}t,$$

and this must hold for any closed smooth curve γ such that z is contained in the area bounded by γ in the complex plane. In particular, we may choose $\gamma(t) = z + e^{2\pi it}$. For this curve, it holds that $|z| - 1 \leq |\gamma(t)| \leq |z| + 1$ and therefore $e^{-b(|z|^2-|\gamma(t)|^2)} \leq e^b$. Also,

$$\left|\frac{\dot\gamma(t)}{(\gamma(t)-z)^{k+1}}\right| = \left|\frac{2\pi i e^{2\pi it}}{e^{(k+1)2\pi it}}\right| = \left|2\pi e^{-k2\pi it}\right| = 1,$$

which shows that

$$\left|f^{(k)}(z)\right| e^{-b\left|z^2\right|} \leq \frac{k!}{2\pi}\|f\|_b \int_0^1 \left|\frac{e^{-b(|z|^2-|\gamma(t)|^2)}}{(\gamma(t)-z)^{k+1}}\dot\gamma(t)\right| \mathrm{d}t \leq k!\,\|f\|_b\, e^b,$$

and therefore

$$\left\|f^{(k)}\right\|_b = \sup_{z\in\mathbb{C}}\left|f^{(k)}(z)\right| e^{-b\left|z^2\right|} \leq k!\,\|f\|_b\, e^b < \infty.$$

This implies that $f^{(k)} \in \mathscr{A}^n_{a+}$.

To prove *(b)*, let $(f_j)_{j\in\mathbb{N}}$ be a bounded sequence in $\mathscr{A}^n_{a+}$ and assume $f_j(z) \to f(z)$ pointwise. Recall that sequence in a Fréchet space is bounded if $\sup_{j\in\mathbb{N}} \|f_j\|_b < \infty$ for all norms $\|\cdot\|_b$, and that a sequence converges to some limit f in the topology of the Fréchet space if $\|f_j - f\|_b \to 0$ as $j \to \infty$. For all $b > a$ and $z \in \mathbb{C}^n$, it is obviously true that

$$\exp\left(-b\sum_{i=1}^n |z_i|^2\right) \leq 1,$$

and therefore $\|f_j - f\|_n \leq \sup_{z\in\mathbb{C}^n} |f(z) - f_j(z)| \to 0$. It remains to show that the pointwise limit f with $f(z) = \lim_{j\to\infty} f_j(z)$ defines an entire function.

Again, we may assume $n = 1$, the generalisation to higher n follows with the same arguments as in Lemma 3.4. Let $z \in \mathbb{C}$. For $k \in \mathbb{N}$, we again use the Cauchy integral formula to write

$$f_j^{(k)}(z) = \frac{k!}{2\pi i} \oint_\gamma \frac{f_j(w)}{(w-z)^{k+1}} \,\mathrm{d}w = \frac{k!}{2\pi i} \int_0^1 \frac{f_j(\gamma(t))}{(\gamma(t)-z)^{k+1}} \dot{\gamma}(t) \,\mathrm{d}t$$

for a suitable curve γ. As z lies in the interior of the area bounded by γ in the complex plane, there is some $d > 0$ such that $\left|(\gamma(t) - z)^{k+1}\right| \geq d$ for all t. Moreover, for any $b > a$, it is clearly true that $\sup\limits_{w\in\gamma} |f_j(w)| \leq \sup\limits_{w\in\gamma} \|f_j\|_b \, e^{b|w|^2}$.

For sufficiently large r, the curve γ is contained in $\{re^{2\pi i t} \colon t \in [0,1]\}$, so that

$$\sup_{w\in\gamma} |f_j(w)| \leq \|f_j\|_b \, e^{br^2} < \infty$$

holds for all j. Denote $M := \sup\limits_{t\in[0,1]} |\dot{\gamma}(t)| < \infty$. It follows that

$$\left| \frac{f_j(\gamma(t))}{(\gamma(t)-z)^{k+1}} \dot{\gamma}(t) \right| \leq \frac{M \|f_j\|_b \, e^{br^2}}{d^{k+1}} \leq \sup_{j\in\mathbb{N}} \|f_j\|_b \frac{M e^{br^2}}{d^{k+1}} \qquad \text{for all } t \in [0,1]\,,$$

where the right side is clearly integrable on $[0,1]$ because the boundedness of the sequence $(f_j)_{j\in\mathbb{N}}$ implies that $\sup\limits_{j\in\mathbb{N}} \|f_j\|_b < \infty$. The dominated convergence theorem can thus be applied to interchange the limit and the integral in

$$f^{(k)}(z) = \lim_{j\to\infty} f_j^{(k)}(z) = \lim_{j\to\infty} \frac{k!}{2\pi i} \oint_\gamma \frac{f_j(w)}{(w-z)^{k+1}} \,\mathrm{d}w,$$

which proves the existence of $f^{(k)}$.

For the converse, first assume that f_j is a bounded sequence converging to 0 in the topology of $\mathscr{A}^n_{a+}$. Then $\|f_j\|_b \to 0$ for all $b > a$ as $j \to \infty$, and therefore

$$|f_j(z)| \leq \exp\left(b \sum_{i=1}^n |z_i|^2 \right) \|f_j\|_b \,.$$

Given that $\|f_j\|_b \to 0$ as $j \to \infty$, pointwise convergence follows.

If $f_j \to f$ and $f \neq 0$, we can replace f_j by $f_j - f$ and apply the same arguments to $f_j - f$.

Finally, *(c)* is a consequence of *(b)*, because for any entire function f, the series of partial sums of the Taylor series converges pointwise to f everywhere. To see that the sequence is also bounded, denote by p_m the partial sum up that includes m terms. The partial sums are polynomials and for any $b > a$ it obviously holds that $\|p_m\|_b \leq \|f\|_b$, and therefore $\sup\limits_{m\in\mathbb{N}} \leq \|p_m\|_b < \infty$, which implies the boundedness of the sequence. □

A direct consequence of Proposition 3.5*(c)* and an important property of the spaces $\mathscr{A}^n_{a+}$ is the next corollary.

Corollary 3.6. *For every $a \geq 0$, the polynomials are dense in $\mathscr{A}^n_{a+}$.*

3.3.2 The case of functions in $\mathscr{A}^n_{a+}$

The first goal of this paragraph is to prove a statement that generalises Proposition 3.1 to be applicable to functions in $\mathscr{A}^n_{a+}$. For entire functions f, g on $\mathbb{C}^n$ with power series $f(z) = \sum_{\boldsymbol{m}\in\mathbb{N}_0^n} \alpha_{\boldsymbol{m}} z^{\boldsymbol{m}}$ and $g(z) = \sum_{\boldsymbol{m}\in\mathbb{N}_0^n} \beta_{\boldsymbol{m}} z^{\boldsymbol{m}}$, conditions are derived that guarantee the well-definedness of the formal power series

$$[f(\partial)g] = \sum_{\boldsymbol{k},\boldsymbol{m}\in\mathbb{N}_0^n} \alpha_{\boldsymbol{k}}\beta_{\boldsymbol{m}}(\partial^{\boldsymbol{k}}/\partial z^{\boldsymbol{k}})z^{\boldsymbol{m}}. \tag{3.15}$$

For $f, g \in \mathscr{A}^n_{a+}, \mathscr{A}^n_{b+}$ respectively, the power series (3.15) is in fact in $\mathscr{A}^n_{c+}$ for some c (which depends on a and b), this is our Corollary 3.8. Finally, by an approximation of (3.15) with polynomials that are nonvanishing whenever $z \in D^n$, we arrive at an analogue of Proposition 3.1, because the nonvanishing in D^n is preserved by the approximation. This analogue is our Proposition 3.10.

The second goal is to show that the function of interest for the Lee-Yang theorem,

$$f(z) = \exp\left(\sum_{i,j=1}^n J_{ij}z_iz_j\right) \qquad \text{with } J_{ij} \geq 0,$$

is in $\mathscr{A}^n_{a+}$. This is done in Proposition 3.11.

We start by identifying conditions under which the power series (3.15) is well-defined.

Proposition 3.7. *Let $a, b > 0, ab < \frac{1}{4}$ and $c > b/(1-4ab)$. Let f, g be entire functions on $\mathbb{C}^n$ with $\|f\|_a, \|g\|_b < \infty$. Then the series*

$$[f(\partial)g] = \sum_{\boldsymbol{k},\boldsymbol{m}} \alpha_{\boldsymbol{k}}\beta_{\boldsymbol{m}}(\partial^{\boldsymbol{k}}/\partial z^{\boldsymbol{k}})z^{\boldsymbol{m}}$$

is absolutely convergent for all z and it defines an entire function such that $\|f(\partial)g\|_c \leq K\|f\|_a\|g\|_b$ for some $K < \infty$ that depends on a, b, c and n, and is independent of f, g.

Proof. Let f be an entire function such that $\|f\|_a < \infty$ and g be an entire function such that $\|g\|_b < \infty$. From Lemma 3.4 recall that

$$|\alpha_{\boldsymbol{k}}| \leq \|f\|_a \prod_{i=1}^n \left(\frac{2ea}{k_i}\right)^{k_i/2},$$

and

$$|\beta_{\boldsymbol{m}}| \leq \|g\|_b \prod_{i=1}^n \left(\frac{2eb}{m_i}\right)^{m_i/2}.$$

There exists a constant $C_1 > 0$ such that $(k/2e)^{k/2} \geq C_1\Gamma((k+1)/2)$ (see for example [Rob55]), it follows that, by combining the products of the constants C_1 for different i to a new constant $C_2 > 0$, the following estimates hold,

$$|\alpha_{\boldsymbol{k}}| \leq C_2 \|f\|_a \prod_{i=1}^{n} \frac{a^{k_i/2}}{A(k_i)}, \tag{3.16}$$

and

$$|\beta_{\boldsymbol{m}}| \leq C_2 \|g\|_b \prod_{i=1}^{n} \frac{b^{m_i/2}}{A(m_i)},$$

where we defined $A(0) = 1$ and $A(2s+1) = A(2s+2) = s!$ for $s \in \mathbb{N}$.

Introduce the entire functions h, l on $\mathbb{C}$, where

$$h(z) = 1 + (z + z^2)\exp(az^2), \quad l(z) = 1 + (z + z^2)\exp(bz^2).$$

The (everywhere absolutely convergent) power series of h is

$$\begin{aligned} 1 + (z+z^2)\exp(az^2) &= 1 + (z+z^2)\sum_{n\in\mathbb{N}_0} \frac{a^n z^{2n}}{n!} = 1 + \sum_{n\in\mathbb{N}_0} \frac{a^n z^{2n+1}}{n!} + \sum_{n\in\mathbb{N}_0} \frac{a^n z^{2n+2}}{n!} \\ &= \frac{z^0}{A(0)} + \frac{1}{a^{1/2}} \sum_{n\in\mathbb{N}_0} \frac{\left(a^{1/2}z\right)^{2n+1}}{A(2n+1)} + \frac{1}{a}\sum_{n\in\mathbb{N}_0} \frac{\left(a^{1/2}z\right)^{2n+2}}{A(2n+2)}, \end{aligned}$$

because $a > 0$ per assumption. For z real and nonnegative, we may bound the power series of h from below,

$$h(z) = \frac{z^0}{A(0)} + \frac{1}{a^{1/2}} \sum_{n\in\mathbb{N}_0} \frac{\left(a^{1/2}z\right)^{2n+1}}{A(2n+1)} + \frac{1}{a}\sum_{n\in\mathbb{N}_0} \frac{\left(a^{1/2}z\right)^{2n+2}}{A(2n+2)} \geq C_3 \sum_{n\in\mathbb{N}_0} \frac{a^{n/2}}{A(n)} z^n,$$

where $C_3 = \min\left\{1, a^{-1/2}, a^{-1}\right\} > 0$. Then

$$\begin{aligned} |f(\partial)g(z)| &\leq \sum_{\boldsymbol{k},\boldsymbol{m}} |\alpha_{\boldsymbol{k}}|\, |\beta_{\boldsymbol{m}}| \left|(\partial/\partial z^{\boldsymbol{k}}) z^{\boldsymbol{m}}\right| |f(\partial)g(z)| \\ &\leq (C_2)^2 \|f\|_a \|g\|_b \sum_{\boldsymbol{k},\boldsymbol{m}} \prod_{i,j=1}^{n} \frac{a^{k_i/2}}{A(k_i)} \frac{b^{m_i/2}}{A(m_i)} \left|(\partial/\partial z_i^{k_i}) z_i^{m_i}\right| \\ &\leq \left(\frac{C_2}{(C_3)^n}\right)^2 \|f\|_a \|g\|_b \prod_{i=1}^{n} \left[h(\partial_w)l\right](w)\Big|_{w=|z_i|}. \end{aligned}$$

Hence the proposition reduces to showing that $[h(\partial)l](z)$ is a well-defined entire function for real and nonnegative $z \in \mathbb{C}$.

The evaluation of the series (3.15) for h, l can be done explicitly. We perform it to show that it represents a well-defined entire function. The constant terms can be ignored. For the evaluation of

$$[h(\partial/\partial z)l](z) = \left(\partial/\partial z + \partial^2/\partial z^2\right)\exp\left(a\partial^2/\partial^2 z\right)\left(z + z^2\right)\exp\left(bz^2\right)$$

we apply the differential operator $\exp\left(a\partial^2/\partial^2 z\right)$ first. For $x \in \mathbb{R}$, we use a standard Gaussian integral to write

$$\exp(ax^2) = C_4 \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{a} + 2tx\right) \mathrm{d}t, \tag{3.17}$$

where $C_4 = (\pi a)^{-1/2}$, and use this formally with $x = \partial/\partial z$. We also use the identity

$$\exp(2t\partial/\partial z) l(z) = l(z + 2t), \tag{3.18}$$

which is proven in Lemma B.1 of the appendix, to write

$$\begin{aligned}
&\exp(a\partial^2/\partial z^2)\left[(z + z^2)\exp(bz^2)\right] \\
&\qquad = C_4 \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{a} + 2t\partial_z\right) \mathrm{d}t \left[(z + z^2)\exp(bz^2)\right] \\
&\qquad = C_4 \int_{-\infty}^{\infty} ((z + 2t) + (z + 2t)^2) \exp\left(-\frac{t^2}{a} + b(z + 2t)^2\right) \mathrm{d}t \\
&\qquad = P_1(a, b; z) \exp(c' z^2).
\end{aligned}$$

In the first step, we inserted (3.17), and in the second step, we used (3.18). The prefactor $c' = b/(1 - 4ab) > 0$ in the exponential comes from the Gaussian integral

$$\begin{aligned}
\int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{a} + b(z + 2t)^2\right) \mathrm{d}t &= \int_{-\infty}^{\infty} \exp\left(-\left(\frac{1}{a} - 4b\right)t^2 + 4bzt + bz^2\right) \mathrm{d}t \\
&= C_4 \exp\left(\frac{bz^2}{\frac{1}{4ab} - 1} + bz^2\right) = C_4 \exp\left(\frac{b}{1 - 4ab} z^2\right),
\end{aligned}$$

and C_4 is a constant that depends on a, b. The convergence of this integral is guaranteed by the assumption $ab < \frac{1}{4}$. The polynomial prefactor in the integrand can be taken care of with partial integration. This yields the polynomial $P_1(a, b; \cdot)$ with coefficients depending on a, b as a prefactor. P_1 is quadratic in z. By simply applying the remaining derivatives $(\partial/\partial z + \partial^2/\partial z^2)$, we get

$$\begin{aligned}
&\left(\partial/\partial z + \partial^2/\partial z^2\right) \exp(a\partial^2/\partial^2 z)\left[(z + z^2)\exp(bz^2)\right] \\
&\qquad = \left(\partial/\partial z + \partial^2/\partial z^2\right) P_1(a, b; z) \exp\left(\frac{b}{1 - 4ab} z^2\right) \\
&\qquad = P_2(a, b; z) \exp(c' z^2),
\end{aligned}$$

where P_2 is a quartic polynomial with coefficients depending on a, b only.

It follows that $\|h(\partial)l\|_\gamma < \infty$ for all $\gamma > c'$. Thus for $c > \frac{b}{1-4ab}$ it must hold that

$$\begin{aligned}\|f(\partial)g(z)\|_c &= \sup_{z\in\mathbb{C}^n} \exp\left(-c\sum_{i=1}^n |z_i|^2\right) |f(\partial)g(z)| \\ &\leq \left(\frac{C_2}{(C_3)^n}\right)^2 \|f\|_a \|g\|_b \prod_{i=1}^n \sup_{z\in\mathbb{C}} \left\{\exp\left(-c|z|^2\right)[h(\partial)l](|z|)\right\} \\ &= K\|f\|_a \|g\|_b ,\end{aligned}$$

where $K = (C_2/(C_3)^n)^2 \|h(\partial)l\|_c^n < \infty$. □

With the previous proposition it is easy to prove that the power series (3.15) for functions f, g in our spaces $\mathscr{A}^n_{a+}, \mathscr{A}^n_{b+}$ respectively is again in such a class of entire functions.

Corollary 3.8. *The map $(f,g) \mapsto f(\partial)g$ is a continuous bilinear map from $\mathscr{A}^n_{a+} \times \mathscr{A}^n_{b+}$ into $\mathscr{A}^n_{b/(1-4ab)+}$ for any $a, b \geq 0$ with $ab < \frac{1}{4}$.*

Proof. The bilinearity of the map is clear. Let $f \in \mathscr{A}^n_{a+}, g \in \mathscr{A}^n_{b+}$ and $c' > b/(1-4ab)$. Pick $a' > a, b' > b$ such that $a'b' < \frac{1}{4}$ and $c' > b'/(1-4a'b')$ (it is possible to check that such a', b', exist).

From Proposition 3.7 we know that

$$\|f(\partial)g\|_{c'} \leq K\|f\|_{a'} \|g\|_{b'} ,$$

so that well-definedness and continuity of the map both follow. □

Next, we define the subset of $\mathscr{A}^n_{a+}$ that is approximable by polynomials which are nonvanishing on some subset $A \subset \mathbb{C}^n$. We will mainly be using $A = D^n$.

Definition 3.9. Let $A \subset \mathbb{C}^n$, denote by $\mathscr{P}^n(A)$ be the set of polynomials on $\mathbb{C}^n$ which are nonvanishing on A and let $\overline{\mathscr{P}^n_{a+}(A)}$ be the closure of $\mathscr{P}^n(A)$ in $\mathscr{A}^n_{a+}$.

The Hurwitz theorem says that the functions in $\overline{\mathscr{P}^n_{a+}(A)}$ must either be nonvanishing in the interior of A or identically zero. This is not necessarily true for all functions in $\mathscr{A}^n_{a+}$ (for an example, see Proposition 3.11), so that in general $\overline{\mathscr{P}^n_{a+}(A)} \subsetneq \mathscr{A}^n_{a+}$.

The following proposition is the desired extension of Proposition 3.1 to functions in $\mathscr{A}^n_{a+}$.

Proposition 3.10. *Let $a, b \geq 0, ab < \frac{1}{4}$, let $f \in \overline{\mathscr{P}^n_{a+}(D^n)}, g \in \mathscr{P}^n_{b+}(D^n)$. Then*

$$h(z) = f(\partial/\partial z)g(z) \in \overline{\mathscr{P}^n_{b/(1-4ab)+}(D^n)}.$$

Proof. For $f, g \in \mathscr{P}^n(D^n)$, Proposition 3.1 immediately implies that $f(\partial/\partial z)g(z) \in \mathscr{P}^n(D^n)$. Let $f \in \overline{\mathscr{P}^n_{a+}(D^n)}, g \in \overline{\mathscr{P}^n_{b+}(D^n)}$. Then there are sequences $(f_k)_{k\in\mathbb{N}}, (g_k)_{k\in\mathbb{N}}$ with $f_k, g_k \in \mathscr{P}^n(D^n)$ and $f_k \to f, g_k \to g$ in the respective spaces. From Proposition 3.1 we know that $f_k(\partial/\partial z)g_k \in \mathscr{P}^n(D^n)$. Finally, $\lim_{k\to\infty} f_k(\partial/\partial z)g_k \in \overline{\mathscr{P}^n_{b/(1-4ab)+}(D^n)}$ follows from Corollary 3.8. □

The next proposition shows that the relevant function for the differential operator (3.4) in our application,

$$f(z) = \exp\left(\sum_{i,j} J_{ij} z_i z_j\right)$$

with $J_{ij} = J_{ji} \geq 0$, is in $\overline{\mathscr{P}^n_{\|J\|+}(D^n)}$.

Proposition 3.11. *Let B be a complex $n \times n$ symmetric matrix and $f(z) = \exp\left(\sum_{i,j} B_{ij} z_i z_j\right)$. Then the following are equivalent:*

(a) $B_{ij} \geq 0$ for all i, j.

(b) $f \in \overline{\mathscr{P}^n_{\|B\|+}(D^n)}$ where $\|B\|$ is the norm of B considered as a bilinear form on $\mathbb{C}^n$ or $\mathbb{R}^n$ equipped with the Euclidean norm.

(c) There exist polynomials $\{P_i\}_{i\in\mathbb{N}}$ in $\mathscr{P}^n(D^n)$ converging pointwise to f.

Proof. We first prove that (a) implies (b). Note that

$$\exp\left(B_{ij} z_i z_j\right) = \lim_{k\to\infty} \left(1 + k^{-1} B_{ij} z_i z_j\right)^k,$$

and that therefore

$$f(z) = \lim_{k\to\infty} \prod_{i,j=1}^{n} \left(1 + k^{-1} B_{ij} z_i z_j\right)^k,$$

We set

$$p_k(z) := \prod_{i,j=1}^{n} \left(1 + k^{-1} B_{ij} z_i z_j\right)^k$$

Since $B_{ij} \geq 0$, the polynomials p_k are all nonzero in D^n. We thus obtain a sequence of polynomials in $\mathscr{P}^n(D^n)$ converging pointwise to f. Using that $(1 + \frac{x}{k})^k \nearrow e^x$ as $k \to \infty$ for $x \geq 0$ yields

$$\begin{aligned} |p_k(z)| &\leq \prod_{i,j=1}^{n} \left(1 + k^{-1} B_{ij} |z_i| |z_j|\right)^k \\ &\leq \exp\left(\sum_{i=1}^{n} B_{ij} |z_i| |z_j|\right) \\ &\leq \exp\left(\|B\| \sum_{i=1}^{n} |z_i|^2\right). \end{aligned}$$

The last inequality follows from the fact that for the bilinear form $(\cdot, B\cdot)$ it holds that $|(z, Bz)| \leq \|B\| \|z\|^2$ where $\|z\|^2 = \sum_{i=1}^{n} |z_i|^2$. This implies the boundedness of the sequence in $\mathscr{A}^n_{\|B\|+}$. The convergence in $\mathscr{A}^n_{\|B\|+}$ is then obtained from Proposition 3.5*(b)*. It follows that $f \in \overline{\mathscr{P}^n_{\|B\|+}(D^n)}$.

It is easy to see that (c) follows from (b) because convergence in $\overline{\mathscr{P}^n_{\|B\|+}(D^n)}$ implies pointwise convergence for any bounded sequence. A sequence of polynomials is obviously bounded (recall A.2), this proves the implication.

It remains to show that (c) implies (a). We fix $(z_2, \cdots, z_n) \in D^{n-1}$, rename $z_1 \to z$ and rearrange f to get a function

$$\hat{f}(z) = C \exp\left(B_{11}z^2 + \left(2\sum_{j=2}^{n} B_{1j}z_j\right) z\right),$$

where C depends on B and on $z_2, \cdots, z_n$, but not on z. Per assumption, there exist polynomials $\{P_i\}_{i\in\mathbb{N}}$ in $\mathscr{P}^n(D^n)$ converging pointwise to f. By fixing $(z_2, \cdots, z_n) \in D^{n-1}$, we can turn the polynomials in n complex variables $\{P_i\}_{i\in\mathbb{N}}$ into polynomials in one complex variable, $\{Q_i\}_{i\in\mathbb{N}}$ in $\mathscr{P}^1(D^1)$, that converge pointwise to $\hat{f}$.

Let $z, z' \in \mathbb{C}$ such that Re $z \geq |\text{Re } z'|$ and Im $z =$ Im z'. Then $Q_i \in \mathscr{P}^1(D^1)$ implies $|Q_i(z)| \geq |Q_i(z')|$ and this inequality is preserved in the limit $\lim_{i\to\infty} Q_i = \hat{f}$. From the form of $\hat{f}$, it is easy to see that this implies $B_{11} \geq 0$ and Re $\sum_{j=2}^{n} B_{1j}z_j \geq 0$. This holds for all $(z_2, \cdots, z_n) \in D^{n-1}$ so $B_{1j} \geq 0$ for $2 \geq j \geq n$. Analogously we can show $B_{ij} \geq 0$ for all i, j. This completes the proof. □

3.3.3 Distribution spaces

From the introduction to the proof, recall that our spin measures μ act as linear functionals on functions in the obvious way,

$$f \mapsto \int_{\mathbb{R}^n} f(\phi)\mathrm{d}\mu(\phi).$$

We can therefore regard the measures as distributions. For our purpose of expressing the partition function (3.1) in terms of these distributions, the action of the distributions on functions of the form $f(\phi) = e^{(\phi,J\phi)+h\phi}$ with $J_{ij} \geq 0$ needs to be well-defined. The distributions will therefore be chosen to be of a certain form. In this section, the corresponding distribution spaces $\mathscr{T}^n_a$ are defined. The Laplace transform is introduced as a tool to transform the measures into entire functions. In Lemma 3.15, we show that the Laplace transform $\hat{T}$ of any distribution $T \in \mathscr{T}^n_a$ is in $\mathscr{A}^n_{1/4a+}$. Finally, Proposition 3.16 proves that the application of a differential operator of the form $f(\partial/\partial z)$, where $f \in \overline{\mathscr{P}^n_{a+}(D^n)}$, to a distribution T with Laplace transform $\hat{T} \in \overline{\mathscr{P}^n_{b+}(D^n)}$ is in $\overline{\mathscr{P}^n_{c+}(D^n)}$ for a suitable c. In particular, it preserves the nonvanishing of $\hat{T}$ in D^n.

Definition 3.12. For $a > 0$, let $\zeta_a(x) := \exp\left(-a\sum_{i=1}^{n} x_i^2\right)$ and $\eta_a(x) = \exp\left(a\sum_{i=1}^{n} x_i^2\right)$. Define $\mathscr{T}^n_a$ to be the space of distributions T on $\mathbb{R}^n$ such that for any function g,

$$T(g) = T_a(\zeta_a g)$$

for a tempered distribution T_a.

Equip $\mathscr{T}_a^n$ with the weak topology generated by the test functions

$$f(x) = \eta_a(x) f_a(x)$$

with $f_a \in \mathscr{S}(\mathbb{R}^n)$ where $\mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz space. That is, a sequence of distributions $T^{(j)} \in \mathscr{T}_a^n$ converges to $T \in \mathscr{T}_a^n$ if and only if the distributions $T_a^{(j)}$ from the above definition converge to T_a in the usual weak topology of the dual space $\mathscr{S}'(\mathbb{R}^n)$. We point out that weak and strong convergence are equivalent in $\mathscr{S}'(\mathbb{R}^n)$ (see Theorem V.26 in [RS72]).

Also, we define $\mathscr{T}_\infty^n = \bigcap_{a>0} \mathscr{T}_a$ with the topology in which a set is open in $\mathscr{T}_\infty^n$ if and only if it is open in the topology of $\mathscr{T}_a^n$ for all $a > 0$.

We will need the following tool to map distributions to functions in a complex variable.

Definition 3.13. Let $a > 0$ and $T \in \mathscr{T}_a^n$. For $z \in \mathbb{C}$ define $\tilde{\eta}_z(x) = e^{zx}$. The function $\tilde{\eta}_z$ is an admissible test function for T because $x \mapsto \tilde{\eta}_z(x)\zeta_a(x) = e^{zx}e^{-ax^2}$ is a Schwartz function. Then the *Laplace transform* is well-defined, and it is defined by

$$T \mapsto \hat{T} \quad \text{where } \hat{T}(z) = T(\tilde{\eta}_z).$$

The Laplace transform is a more general concept (see for instance [Dav02]), but for the purpose of this thesis it is sufficient to define it only for distributions in $\mathscr{T}_a^n$. The relevant properties of the Laplace transform for us are discussed in Lemma 3.15. For the proof of Lemma 3.15, we need the following lemma.

Lemma 3.14. *Let $\boldsymbol{k} \in \mathbb{N}_0^n$ with $|\boldsymbol{k}| = N$, $M \in \mathbb{N}_0$, $z \in \mathbb{C}^n$ and $f_z(x) = \exp\left(zx - ax^2\right)$. Then there is a constant C (dependent on M, N) such that the following inequality holds*

$$\sup_{x\in\mathbb{R}^n} \left(1 + |x|^M\right) \left|\partial_x^{\boldsymbol{k}} f_z(x)\right| \leq C \left(1 + |z|^{N+M+1}\right) \exp\left(\frac{|z|^2}{4a}\right).$$

The proof involves lengthy calculations that are not directly relevant to the technical procedure for proving the Lee-Yang theorem. Therefore, we defer it to Section C in the appendix.

The next lemma states an important property of the Laplace transform.

Lemma 3.15. *Let $0 < a \leq \infty$. Then the Laplace transform is a sequentially continuous linear map of $\mathscr{T}_a^n$ into $\mathscr{A}_{1/4a+}^n$. Equivalently, if a sequence $T^{(j)}$ converges to T in $\mathscr{T}_a^n$, then $\hat{T}^{(j)}$ converges to $\hat{T}$ in $\mathscr{A}_{1/4a+}^n$.*

Proof. The linearity of the map $T \mapsto \hat{T}$ follows from the linearity of distributions.

To prove sequential continuity, assume first that $0 < a < \infty$. Then for $T \in \mathscr{T}_a^n$ there is a tempered distribution $T_a \in \mathscr{S}'(\mathbb{R}^n)$ such that $\hat{T}(z) = T_a(f_z)$ where $f_z(x) = \exp(zx - ax^2)$. Since $T_a \in \mathscr{S}'(\mathbb{R}^n)$, it follows directly from Theorem V.10 in [RS72] or Theorem 7.2.1 in [CMÖ21] that there exist $M, N \in \mathbb{N}_0$ such that

$$|T_a(f_z)| \leq K \sup_{x\in\mathbb{R}^n} \left(1 + |x|^M\right) \sum_{|\boldsymbol{k}|\leq N} \left|\partial^{\boldsymbol{k}} f_z(x)\right| \tag{3.19}$$

for some K dependent on M, N. From Lemma 3.14 we know that the inequality

$$\left(1+|x|^M\right)\left|\partial^{\boldsymbol{k}} f_z(x)\right| \leq C\left(1+|z|^{|\boldsymbol{k}|+M+1}\right)\exp\left(\frac{|z|^2}{4a}\right) \tag{3.20}$$

holds for a suitable constant C depending on $M, |\boldsymbol{k}|$ and a. Hence $\hat{T} \in \mathscr{A}^n_{1/4a+}$.

Now consider a sequence $(T^{(j)})_{j\in\mathbb{N}}$ in $\mathscr{T}^n_a$ that converges to $T \in \mathscr{T}^n_a$. Let $\left(T^{(j)}_a\right)_{j\in\mathbb{N}}, T_a \in \mathscr{S}'(\mathbb{R}^n)$ such that $\hat{T}^{(j)}(z) = T^{(j)}_a(f_z)$ and $\hat{T}^{(j)}(z) = T_a(f_z)$.

For $z \in \mathbb{C}^n$, let $g_{z,b}(x) = \exp\left(zx - ax^2 - b|z|^2\right) = \exp\left(-b|z|^2\right) f_z(x)$. We will use that for all b, the linearity of T implies that

$$\exp\left(-b|z|^2\right)\hat{T}(z) = T_a(g_{z,b}). \tag{3.21}$$

In particular, the functions $g_{z,b}$ are admissible test functions for $T^{(j)}_a$ and T_a.

To prove that $\hat{T}^{(j)} \to \hat{T}$ in the topology of $\mathscr{A}^n_{1/4a+}$, we need to show that

$$\sup_{z\in\mathbb{C}^n}\left\{\left|T^{(j)}_a(g_{z,b}) - T_a(g_{z,b})\right|\right\} \to 0 \quad \text{for all } b > \frac{1}{4a}. \tag{3.22}$$

We therefore fix $b > \frac{1}{4a}$. To prove (3.22), we use the equivalence of weak and strong convergence in $\mathscr{S}'(\mathbb{R}^n)$, because this implies that $T^{(j)}_a(g_{z,b}) \to T_a(g_{z,b})$. The inequality (3.19) together with the (yet to be proven) fact that $\{g_{z,b}\}_{z\in\mathbb{C}^n}$ is a bounded family in $\mathscr{S}(\mathbb{R}^n)$ shows that the convergence of $\hat{T}^{(j)}(z)$ to $\hat{T}(z)$ is uniform. This proves (3.22). It follows that $\hat{T}^{(j)} \to \hat{T}$ in the topology of $\mathscr{A}^n_{1/4a+}$.

To prove the boundedness of $\{g_{z,b}\}_{z\in\mathbb{C}^n}$, note that equipped with the topology induced by the seminorms $\|f\|_{\boldsymbol{k},\boldsymbol{m}} := \sup_{x\in\mathbb{R}^n}\left\{\left|x^{\boldsymbol{k}}(\partial^{\boldsymbol{m}} f)(x)\right|,\ \boldsymbol{k},\boldsymbol{m} \in \mathbb{N}^n_0\right\}$, the space $\mathscr{S}(\mathbb{R}^n)$ is a Fréchet space (see pp. 29 in [Str03]). We know that a family in a Fréchet space is bounded when all its seminorms are bounded. We calculate

$$g_{z,b}(x) = f_z(x)e^{-b|z|^2}, \quad \partial^{\boldsymbol{k}}_x g_{z,b}(x) = e^{-b|z|^2}\partial^{\boldsymbol{k}}_x f_z(x).$$

It is easy to see that $\left|x^{\boldsymbol{k}}\right| = \prod_{i=1}^n |x_i|^{k_i} \leq \prod_{i=1}^n |x|^{k_i} = |x|^{|\boldsymbol{k}|}$ with $|\boldsymbol{k}| = \sum_{i=1}^n k_i$. For any $\boldsymbol{k},\boldsymbol{m} \in \mathbb{N}^n_0$ with $M = |\boldsymbol{k}|$ and $N = |\boldsymbol{m}|$, the inequality (3.20) implies

$$\begin{aligned}\left|x^{\boldsymbol{k}}\partial^{\boldsymbol{m}} g_{z,b}(x)\right| &\leq \left(1+|x|^M\right)|\partial^{\boldsymbol{m}} g_{z,b}(x)| \\ &= \left(1+|x|^M\right)|\partial^{\boldsymbol{m}} f_z(x)|e^{-b|z|^2} \\ &\leq C\left(1+|z|^{N+M+1}\right)\exp\left(-\left(b-\frac{1}{4a}\right)|z|^2\right).\end{aligned}$$

Per assumption, $b > \frac{1}{4a}$, hence the prefactor in front of $|z|^2$ in the exponential is negative. This implies that for all $z \in \mathbb{C}^n$, the seminorms $\|g_{z,b}\|_{\boldsymbol{k},\boldsymbol{m}} = \sup_{x\in\mathbb{R}^n} \left\{ \left| x^{\boldsymbol{m}} \partial^{\boldsymbol{k}} g_{z,b}(x) \right| \right\}$ are bounded by

$$\|g_{z,b}\|_{\boldsymbol{k},\boldsymbol{m}} \leq \sup_{z\in\mathbb{C}^n} C\left(1+|z|^{N+M+1}\right) \exp\left(-\left(b-\frac{1}{4a}\right)|z|^2\right) < \infty,$$

and therefore, $\{g_{z,b}\}_{z\in\mathbb{C}^n}$ is a bounded family in $\mathscr{S}(\mathbb{R}^n)$.

The case $a=\infty$ follows immediately from the previous since convergence in $\mathscr{T}^n_\infty$ respectively $\mathscr{A}^n_{0+}$ is equivalent to convergence in $\mathscr{T}^n_a$ for all $a<\infty$ respectively convergence in $\mathscr{A}^n_{\varepsilon+}$ for all $\varepsilon > 0$. □

This lemma allows us to derive an important result about the multiplication of the Laplace transform of a measure with functions in $\overline{\mathscr{P}^n_{\alpha+}(D^n)}$.

Proposition 3.16. *Let $0 \leq \alpha < \beta \leq \infty$, let T be a distribution in $\mathscr{T}^n_\beta$ whose Laplace transform $\hat{T}$ lies in $\overline{\mathscr{P}^n_{1/4\beta+}(D^n)}$ and let $f \in \overline{\mathscr{P}^n_{\alpha+}(D^n)}$. Then, for every $\gamma < \beta-\alpha$ (and for $\gamma=\infty$ if $\beta=\infty$), the distribution fT lies in $\mathscr{T}^n_\gamma$ and its Laplace transform $\widehat{fT}$ lies in $\overline{\mathscr{P}^n_{1/4\gamma+}(D^n)}$.*

Proof. Assume that $\beta, \gamma < \infty$ first. We begin by proving that $fT \in \mathscr{T}^n_\gamma$. For $a>0$, let $\zeta_a(x) := e^{-ax^2}$ and $\eta_a(x) = e^{ax^2}$, where $x\in\mathbb{R}^n$ and x^2 is to be understood as the scalar product $x^2 = \sum_{i=1}^n x_i^2$. Let $f \in \overline{\mathscr{P}^n_{\alpha+}(D^n)}$ and $T \in \mathscr{T}^n_\beta$. Let $g = \eta_\gamma g_\gamma$ be a test function with $g_\gamma \in \mathscr{S}(\mathbb{R}^n)$. The statement will be proven by showing that $f\zeta_\beta\eta_\gamma g_\gamma$, where f is regarded as a function of a real variable in the obvious way, is a Schwartz function. Given that we can write $T(\varphi) = T_\beta(\zeta_\beta\varphi)$ for some tempered distribution T_β and any test function φ, the claim will follow from the properties of tempered distributions.

It is therefore sufficient to show that $f\zeta_\beta\eta_\gamma g_\gamma \in \mathscr{S}(\mathbb{R}^n)$. Recall that $f \in \overline{\mathscr{P}^n_{\alpha+}(D^n)}$ implies $|f(z)| \leq e^{b|z|^2}\|f\|_b$ for all $b>a$ and for all $z\in\mathbb{C}^n$. In particular, as $\beta-\gamma>\alpha$, it follows that

$$e^{-(\beta-\gamma)|z|^2}|f(z)| = \zeta_\beta(|z|)\eta_\gamma(|z|)\,|f(z)| \leq \|f\|_{\beta-\gamma} < \infty \quad \text{for all } z\in\mathbb{C}^n. \tag{3.23}$$

Proposition 3.5*(a)* guarantees that this also holds for all derivatives of f. Let $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathbb{N}^n_0$, then

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}\{|x|^{\boldsymbol{m}_1}|\partial_x^{\boldsymbol{m}_2}(f\zeta_\beta\eta_\gamma g_\gamma)(x)|\} &= \sup_{x\in\mathbb{R}^n}\left\{|x|^{\boldsymbol{m}_1}|\sum_{\boldsymbol{k}\leq\boldsymbol{m}_2} p_{\boldsymbol{k}}(x)\zeta_\beta\eta_\gamma\partial_x^{\boldsymbol{k}}f(x)\partial_x^{\boldsymbol{l}(\boldsymbol{k})}g_\gamma(x)|\right\}\\ &\leq M_1\sup_{x\in\mathbb{R}^n}\left\{|x|^{\boldsymbol{m}_1}|\sum_{\boldsymbol{k}\leq\boldsymbol{m}_2} p_{\boldsymbol{k}}(x)\partial_x^{\boldsymbol{l}(\boldsymbol{k})}g_\gamma(x)|\right\} < \infty,\end{aligned}$$

where we defined $M_1 := \max_{\boldsymbol{k}\leq\boldsymbol{m}_2}\left\{\sup_{z\in\mathbb{C}^n}\left\{\zeta_\beta(|z|)\eta_\gamma(|z|)\left|\partial^{\boldsymbol{k}}f(z)\right|\right\}\right\}$ and $M_1<\infty$ by (3.23). The multi-index inequalities are to be understood componentwise, $p_{\boldsymbol{k}}$ are the multivariable polynomials of degree at most $|\boldsymbol{m}_2|$ in each variable x_i that are obtained from differentiating $\zeta_\beta\eta_\gamma$, and $\boldsymbol{l}(\boldsymbol{k}) \leq \boldsymbol{m}$ are the corresponding multiindices that result from the application of the product rule. The supremum is finite because $\sum_{\boldsymbol{k}\leq\boldsymbol{m}_2} p_{\boldsymbol{k}}\partial_x^{\boldsymbol{l}(\boldsymbol{k})}g_\gamma$ is a Schwartz function (recall that the Schwartz space is closed under differentiation and under multiplication with polynomials). It follows that

$$fT(g) = T_\beta(f\zeta_\beta\eta_\gamma g_\gamma)$$

is well-defined. This proves $fT \in \mathscr{T}_\gamma^n$.

To prove the statement about the Laplace transform, assume first that $f \in \mathscr{P}^n(D^n)$. Then it is easy to see that

$$\widehat{fT}(z) = T(f(x)e^{zx}) = T(f(\partial/\partial z)e^{zx}) = f(\partial/\partial z)\hat{T}(z),$$

so by Proposition 3.10 with $a = \alpha, b = 1/4\beta$ we have $\widehat{fT} \in \overline{\mathscr{P}^n_{1/4(\beta-\alpha)+}(D^n)} \subset \overline{\mathscr{P}^n_{1/4\gamma+}(D^n)}$ with the obvious inclusions of the spaces.

For general $f \in \overline{\mathscr{P}^n_{\alpha+}(D^n)}$ let $\{f_j\}$ be a sequence in $\mathscr{P}^n(D^n)$ converging to $f \in \mathscr{A}^n_{a+}$. Note that this implies not only $\|f_j - f\|_b \to 0$ for all $b > \alpha$, but also $\left\|\partial^{\boldsymbol{k}} f_j - \partial^{\boldsymbol{k}} f\right\|_b \to 0$ for all $\boldsymbol{k} \in \mathbb{N}_0^n$.

Then, as in the proof of Lemma 3.15, we know that

$$\begin{aligned}|T_\beta(f\zeta_\beta\eta_\gamma g_\gamma)| &\leq K \sup_{x\in\mathbb{R}^n}\left(1+|x|^M\right)\sum_{|\boldsymbol{k}|\leq N}\left|\partial^{\boldsymbol{k}}(f\zeta_\beta\eta_\gamma g_\gamma)(x)\right| \\ &\leq KM_2 \max_{|\boldsymbol{k}|\leq N}\left\{\left\|\partial^{\boldsymbol{k}} f\right\|_{\beta-\gamma}\right\},\end{aligned}$$

for some $M, N \in \mathbb{N}, K > 0$, and $M_2 = \sup_{x\in\mathbb{R}^n}\left\{(1+|x|^M)|\sum_{|\boldsymbol{k}|\leq N} p_{\boldsymbol{k}}(x)\partial_x^{\boldsymbol{l}(\boldsymbol{k})} g_\gamma(x)|\right\} < \infty$, with similar arguments as used above.

This shows that there is some a K_2 and a multiindex $\boldsymbol{k} \in \mathbb{N}_0^n$ such that for any test function g,

$$|f_jT(g) - fT(g)| \leq K_2\left\|\partial^{\boldsymbol{k}} f_j - \partial^{\boldsymbol{k}} f\right\|_{\beta-\gamma} \to 0 \quad \text{as } j \to \infty,$$

so $\{f_jT\}$ converges to fT in the topology of $\mathscr{T}_\gamma^n$.

By Lemma 3.15, $\{\widehat{f_jT}\}$ converges to $\widehat{fT}$ in $\mathscr{A}^n_{1/4\gamma+}$. As $\overline{\mathscr{P}^n_{1/4\gamma+}(D^n)}$ is closed in $\mathscr{A}^n_{1/4\gamma+}$, the limit $\hat{fT} = \lim_{j\to\infty} \hat{f_jT}$ is in $\overline{\mathscr{P}^n_{1/4\gamma+}(D^n)}$, because any closed subset of a Fréchet space is sequentially closed (see Proposition A.4).

The case $\beta = \infty$ (and possibly also $\gamma = \infty$) follows immediately from the previous since convergence in $\mathscr{T}^n_\infty$ respectively $\mathscr{A}^n_{0+}$ is equivalent to convergence in $\mathscr{T}^n_a$ for all $a < \infty$ respectively convergence in $\mathscr{A}^n_{\varepsilon+}$ for all $\varepsilon > 0$. □

3.4 One-component models

In this section, we show that the previously derived propositions can be applied to the situation introduced in Section 3.1.2. We prove the main result of this thesis, a Lee-Yang theorem for general one-component ferromagnets. This is Corollary 3.19.

Recall that we can write the partition function as

$$\mathcal{Z}(h_1, \cdots, h_n) = \int_{\mathbb{R}^n} \exp\left(\sum_{i,j=1}^{n} J_{ij} \frac{\partial}{\partial h_i} \frac{\partial}{\partial h_j}\right) \exp\left(\sum_{i=1}^{n} h_i \phi_i\right) \prod_{i=1}^{n} \mathrm{d}\nu_i(\phi_i).$$

In order to describe the partition function using the terminology developed in Section 3.3, set

$$f(\phi) = \exp\left(\sum_{i,j=1}^{n} J_{ij} \phi_i \phi_j\right), \qquad \mathrm{d}\mu_0(\phi) = \prod_{i=1}^{n} \mathrm{d}\nu_i(\phi_i).$$

With the previously developed methods we see that we can write the free partition function as the Laplace transform of the measure μ_0,

$$\mathcal{Z}_0(h_1, \cdots, h_n) = \int_{\mathbb{R}^n} \exp\left(\sum_{i=1}^{n} h_i \phi_i\right) \prod_{i=1}^{n} \mathrm{d}\nu_i(\phi_i) = \hat{\mu}_0(h_1, \cdots, h_n),$$

and the full partition function as the Laplace transform of $f\mu_0$,

$$\mathcal{Z}(h_1, \cdots, h_n) = \widehat{f\mu_0}(h_1, \cdots, h_n).$$

Propositions 3.11 and 3.16 imply that for $\widehat{f\mu_0}$ to be nonvanishing on D^n, it is sufficient that $J_{ij} \geq 0$ holds for all i, j and that $\hat{\mu}_0 \in \overline{\mathscr{P}^n_{c+}(D^n)}$ for $c < \frac{1}{4\|J\|}$ (or any nonnegative c if $\|J\| = 0$). These are essentially the considerations which enter into the proof of the Lee-Yang theorem below.

We first introduce some terminology to make the statements more precise.

Definition 3.17. A finite positive measure μ on $\mathbb{R}^n (\mu \not\equiv 0)$ is said to have the *Lee-Yang property* (with falloff β) if $\mu \in \mathscr{T}^n_\beta$ and $\hat{\mu} \in \overline{\mathscr{P}^n_{1/4\beta+}(D^n)}$.

Since $\mu \not\equiv 0$ implies $\hat{\mu} \not\equiv 0$, it follows that $\hat{\mu}(z) \neq 0$ for Re $z > 0$, this is usually the conclusion of the Lee-Yang theorem. The statement here is even stronger because it entails that $\hat{\mu}$ is approximable by polynomials with this property.

Theorem 3.18. *Let μ_0 have the Lee-Yang property with falloff β. Let $f \in \overline{\mathscr{P}^n_{\alpha+}(D^n)}$ for some $\alpha < \beta$ be nonnegative on the support of μ_0, and strictly positive on a set of nonzero μ_0-measure. Then $\mu = f\mu_0$ has the Lee-Yang property with falloff γ, for every $\gamma < \beta - \alpha$ (and $\gamma = \infty$ if $\beta = \infty$). In particular, we can take*

$$f(\phi) = \exp\left(\sum_{i,j=1}^{n} J_{ij} \phi_i \phi_j\right), \tag{3.24}$$

where $J_{ij} \geq 0$, provided that $\alpha = \|J\| < \beta$, where $\|J\|$ is the norm of J considered as a bilinear form on $\mathbb{R}^n$ equipped with the Euclidean norm.

Proof. Proposition 3.16 states that $fT \in \mathscr{T}^n_\gamma$ and $\widehat{fT} \in \overline{\mathscr{P}^n_{1/4\gamma+}(D^n)}$ whenever $f \in \overline{\mathscr{P}^n_{\alpha+}(D^n)}$. The positivity condition on f assures that $\mu \geq 0$ and $\mu \not\equiv 0$. Proposition 3.11 implies that (3.24) is in $\overline{\mathscr{P}^n_{\|J\|+}(D^n)}$. Furthermore, as (3.24) is strictly positive on $\mathbb{C}^n$, it fulfils the requirements of the theorem. □

From this theorem, the Lee-Yang theorem for one-component ferromagnets follows.

Corollary 3.19. *Let $\{\nu_i\}_{i=1}^n$ be measures on $\mathbb{R}^1$, each having the Lee-Yang property with falloff β. Let J be a symmetric $n \times n$ matrix with nonnegative entries and $\|J\| < \beta$. Then the measure μ on $\mathbb{R}^n$ given by*

$$\mathrm{d}\mu(\phi) = \exp\left(\sum_{i,j=1}^{n} J_{ij}\phi_i\phi_j\right)\prod_{i=1}^{n}\mathrm{d}\nu_i(\phi_i)$$

has the Lee-Yang property with falloff γ for every $\gamma < \beta - \|J\|$ (and $\gamma = \infty$ if $\beta = \infty$). In particular, we can let each measure ν_i be an even measure satisfying

$$\int e^{h\phi}\mathrm{d}\nu_i(\phi) \neq 0 \text{ for } \mathrm{Re}\ h \neq 0. \tag{3.25}$$

Proof. The action of the measures ν_i on any test function φ_i can be written as $\nu_i(\varphi_i) = T_{\beta,i}(\zeta_{\beta,i}\varphi_i)$ where $\zeta_{\beta,i}(x_i) = e^{-\beta x_i^2}$ and $T_{\beta(x),i}$ is some tempered distribution for each i. The additional index i has been added to $\zeta_{\beta,i}$ to indicate that it acts on the variable x_i. Then, for any test function φ, it holds that

$$\left(\prod_{i=1}^{n}\nu_i\right)(\varphi) = \left(\prod_{i=1}^{n}T_{\beta,i}\right)\left(\prod_{i=1}^{n}\zeta_{\beta,i}\varphi\right),$$

where $\zeta_\beta(x) = \exp\left(-\beta\sum_{i=1}^{n}x_i^2\right)$ and $\prod_{i=1}^{n}T_{\beta,i} \in \mathscr{S}'(\mathbb{R}^n)$. This proves $\prod_{i=1}^{n}\nu_i \in \mathscr{T}_\beta^n$. Theorem 3.18 implies that $\mu = f\prod_{i=1}^{n}\nu_i$ has the Lee-Yang property with falloff γ for every $\gamma < \beta - \|J\|$ (and $\gamma = \infty$ if $\beta = \infty$).

The last remark is a consequence of Lemma 3.21. □

For the proof of Lemma 3.21, we need the following lemma about the Laplace transform of a measure.

Lemma 3.20. *Let ν be a measure on $\mathbb{R}$. Then the Laplace transform $\hat{\nu}$ is either nonvanishing in at least one of the limits $h \to \pm\infty$ or identically zero.*

Proof. To see that, assume

$$\int e^{a|\phi|}\mathrm{d}\nu(\phi) < \infty \qquad \text{for all } a \geq 0. \tag{3.26}$$

For $h \in \mathbb{R}$, define $L(h) = \int_{\mathbb{R}} e^{h\phi}\mathrm{d}\nu(\phi)$. We claim that $L(h)$ is either independent of h or diverges in at least one of the limits $h \to \pm\infty$. The claim implies the statement of the lemma because for $h \in \mathbb{R}$ it is $L(h) = \hat{\nu}(h)$.

To prove the claim, let $z = h + ik$ for $h, k \in \mathbb{R}$. Then, for $\phi \in \mathbb{R}$ it holds that $\left|e^{z\phi}\right| = e^{h\phi}$, so the assumption (3.26) guarantees the well-definedness of

$$\hat{\nu}(z) = \int_{\mathbb{R}} e^{z\phi} \mathrm{d}\nu(\phi) = \int_{\mathbb{R}} e^{h\phi} \cos(k\phi) \mathrm{d}\nu(\phi) + i \int_{\mathbb{R}} e^{h\phi} \sin(k\phi) \mathrm{d}\nu(\phi).$$

Moreover, the derivative

$$\hat{\nu}'(z) = \int_{\mathbb{R}} \phi e^{z\phi} \mathrm{d}\nu(\phi)$$

is well-defined at every $z \in \mathbb{C}$. To see that, note first that $|\phi| \leq e^{|\phi|}$, this implies

$$\left|\hat{\nu}'(z)\right| \leq \int_{\mathbb{R}} e^{(|z|+1)|\phi|} \mathrm{d}\nu(\phi).$$

The assumption (3.26) and the dominated convergence theorem finish the proof of the existence of $\hat{\nu}'$. Hence $\hat{\nu}$ is an entire function and the Liouville theorem (Theorem 3.7 in [FB06]) applies. The function $\hat{\nu}$ is therefore either constant or unbounded. If it is not constant, the inequality

$$|\hat{\nu}(z)| \leq \int_{\mathbb{R}} e^{h\phi} \mathrm{d}\nu(\phi) = L(h)$$

implies that the right-hand side cannot be bounded on $\mathbb{R}$, so that the claim is proven. □

Lemma 3.21. *Let ν be an even measure in $\mathscr{T}^1_\beta$ satisfying $\int e^{h\phi} \mathrm{d}\nu_i(\phi) \neq 0$ for $\operatorname{Re} h \neq 0$. Then $\hat{\nu} \in \overline{\mathscr{P}^1_{1/4\beta+}(D^1)}$ and*

$$\hat{\nu}(h) = K e^{bh^2} \prod_j \left(1 + \frac{h^2}{\alpha_j^2}\right)$$

with $K > 0$ as well as $0 \leq b \leq 1/4\beta$ and $0 < \alpha_1 \leq \alpha_2 \leq \cdots$, where $\sum\limits_{j\in\mathbb{N}} \alpha_j^{-2} < \infty$ and the sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ may be empty, finite or infinite.

Proof. First note that $\hat{\nu}$ is even because

$$\hat{\nu}(-h) = \int e^{-hx} \mathrm{d}\nu(x) = \int e^{h(-x)} \mathrm{d}\nu(-x) = \int e^{hy} \mathrm{d}\nu(y) = \hat{\nu}(h).$$

Per assumption, $\hat{\nu}$ has no zeros with $\operatorname{Re} z > 0$ and, as a consequence of its evenness, $\hat{\nu}$ cannot have any zeros with $\operatorname{Re} z < 0$, thus $\hat{\nu}$ has only purely imaginary zeros. In particular, they must appear in complex conjugate pairs.

Use the Hadamard theorem (Theorem I.10.13 in [Lev64]) to write

$$\hat{\nu}(z) = z^m e^{Q(z)} \prod_{n\in\mathbb{N}} \left(1 - \frac{z}{a_n}\right) \prod_{k=1}^{p} \frac{e^k}{k}, \tag{3.27}$$

where m is the multiplicity of the root at the origin, Q is a polynomial, a_n are the nonzero roots of $\hat{\nu}$, and p is smallest non-negative integer such that the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$ converges. The sequence $(a_n)_{n\in\mathbb{N}}$ may be empty, finite or infinite.

Because $\hat{\nu}$ is in $\overline{\mathscr{P}^1_{1/4\beta+}(D^1)}$, it is of exponential order at most 2. Hence the polynomial Q must be of the form

$$Q(z) = bz^2 + cz + d.$$

Given that $\hat{\nu}$ is even, it must hold that $c = 0$. Moreover, since $\hat{\nu} \in \mathbb{R}$ for real h, it follows that b must be real. Lemma 3.20 shows that $\hat{\nu}$ must be nonvanishing in at least one of the limits $z \to \pm\infty$ due to the assumption $\hat{\nu} \in \overline{\mathscr{P}^1_{1/4\beta+}(D^1)}$. This implies $b \geq 0$. Finally, d can be absorbed into a constant prefactor, and we can see that $m = 0$ because $\hat{\nu}(0) = \int \mathrm{d}\nu \neq 0$ is implied by $\nu \not\equiv 0$.

The last product in (3.27) is a constant and will therefore also be absorbed into a constant prefactor.

The roots are purely imaginary and appear in complex conjugated pairs, therefore we write $a_{n_{1,2}} := \pm i\alpha_n$ with $\alpha_n \in \mathbb{R} \setminus \{0\}$. This yields the representation

$$\hat{\nu}(z) = Ke^{bz^2} \prod_{n\in\mathbb{N}} \left(1 + \frac{z^2}{\alpha_n^2}\right). \tag{3.28}$$

A sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ approximating $\hat{\nu}$ can be obtained by cutting the infinite product in (3.28) off at a finite $n_{\max} = n$. These polynomials are bounded in absolute value by $\hat{\nu}(|h|)$,

$$\left| Ke^{bz^2} \prod_{n=1}^{n_{\max}} \left(1 + \frac{z^2}{\alpha_n^2}\right) \right| \leq Ke^{b|z|^2} \prod_{n=1}^{n_{\max}} \left(1 + \frac{|z|^2}{\alpha_n^2}\right) = \hat{\nu}(|h|),$$

so that for all $\gamma > \frac{1}{4\beta}$ it holds that $\sup_{n\in\mathbb{N}} \|p_n\|_\gamma \leq \|\hat{\nu}\|_\gamma$. The sequence is therefore bounded and pointwise convergent to $\hat{\nu}$. By Proposition 3.5*(b)*, convergence in $\mathscr{A}^1_{1/4\beta+}$ follows. □

3.5 Two-component models

In this section, we derive a Lee-Yang theorem for general two-component ferromagnets. We show that, by a change of variables, the partition function of a two-component ferromagnet can be traced back to the partition function of a one-component ferromagnet with twice as many spins, such that the Lee-Yang theorem immediately follows from the corresponding statements from Section 3.4. It then only remains to express the result in terms of the initial variables in order to determine the conditions under which a Lee-Yang theorem for general two-component ferromagnets holds.

The Hamiltonian of a two-component ferromagnet is given by

$$H = \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{2} J_{ij}^{(\alpha\beta)} \phi_i^{(\alpha)} \phi_j^{(\beta)} + \sum_{i=1}^{n} \sum_{\alpha=1}^{2} h_i^{(\alpha)} \phi_i^{(\alpha)},$$

where the upper index (in Greek letters) denotes the spin component and the lower index (in Roman letters) numbers the spins on the lattice, as usual. We will use bold variables when referring to a two-component magnetic field, $\boldsymbol{h} = (h^{(1)}, h^{(2)})$ or a two-component spin, $\boldsymbol{\phi} = (\phi^{(1)}, \phi^{(2)})$.

We assume that there is no coupling between the different spin components, that is,

$$J_{ij}^{(12)} = J_{ij}^{(21)} = 0 \quad \text{for all } i, j.$$

The partition function is

$$\mathcal{Z}(h_1^{(1)}, \cdots, h_n^{(2)}) = \int \exp\left(\sum_{i,j=1}^{n}\sum_{\alpha=1}^{2} J_{ij}^{(\alpha\alpha)}\phi_i^{(\alpha)}\phi_j^{(\alpha)} + \sum_{i=1}^{n}\sum_{\alpha=1}^{2} h_i^{(\alpha)}\phi_i^{(\alpha)}\right)\prod_{i=1}^{n}\mathrm{d}\nu_i(\boldsymbol{\phi}_i). \tag{3.29}$$

For a general N-component ferromagnet, the partition function can be defined analogously by replacing the sum over $\alpha = 1, 2$ by a sum over $\alpha = 1, \cdots, N$. In the following, we first derive some statements that hold for any N-component ferromagnet ($N \geq 2$) and restrict the discussion to the case $N = 2$, for which we can derive a Lee-Yang theorem, later.

We assume that the system is invariant under rotations, meaning that the measures ν_i are all rotationally invariant. If a Lee-Yang theorem holds, it must thus hold regardless of the orientation of the magnetic field, so we can let the magnetic field point in the 1-direction. It is therefore natural to replace the assumption (3.26) by

$$\int e^{h\phi^{(1)}}\mathrm{d}\nu(\boldsymbol{\phi}) \neq 0 \text{ for Re } h \neq 0.$$

We show in Proposition 3.22 that this results in the zero-free region

$$L_N^{\pm} = \left\{\boldsymbol{h}\colon \pm \text{Re } h^{(1)} > \left[\sum_{\alpha=2}^{N}(\text{Im } h^{(\alpha)})^2\right]^{1/2}\right\}. \tag{3.30}$$

For a rotationally invariant measure ν on $\mathbb{R}^N$ that describes the distribution of a general N-component spin, we can derive the following statements.

Proposition 3.22. *Let $N \geq 2$ and $\nu \in \mathscr{T}_\beta^N$ be a rotationally invariant measure on $\mathbb{R}^N$, satisfying*

$$\int e^{h\phi^{(1)}}\mathrm{d}\nu_i(\boldsymbol{\phi}) \neq 0 \textit{ for } \text{Re } h \neq 0. \tag{3.31}$$

Then the Laplace transform $\hat{\nu}$ is of the form

$$\hat{\nu}(\boldsymbol{h}) = F\left(\sum_{\alpha=1}^{N} {h^{(\alpha)}}^2\right), \tag{3.32}$$

where F is an entire function of order at most 1 with only real negative zeros. More precisely,

$$\hat{\nu}(\boldsymbol{h}) = Ke^{b\zeta}\prod_j\left(1 + \frac{\zeta}{\alpha_j^2}\right) \tag{3.33}$$

with $\zeta = \sum_{\alpha=1}^{N} {h^{(\alpha)}}^2, K > 0, 0 \leq b \leq 1/4\beta$ and $0 < \alpha_1 \leq \alpha_2 \leq \cdots$ with $\sum_j \alpha_j^{-2} < \infty$. The se-

quence $(\alpha_j)_{j\in\mathbb{N}}$ *may be empty, finite or infinite.*

Moreover, $\hat{\nu} \in \overline{\mathscr{P}^N_{1/4\beta+}(L_N)}$*, where*

$$L_N^{\pm} = \left\{ \boldsymbol{h} \colon \pm \operatorname{Re} h^{(1)} > \left[\sum_{\alpha=2}^{N} (\operatorname{Im} h^{(\alpha)})^2 \right]^{1/2} \right\} \quad \text{and} \quad L_N := L_N^+ \cup L_N^-. \tag{3.34}$$

Before proving the proposition, we emphasise that the condition (3.31) says that the projection of ν onto the first coordinate has the Lee-Yang property as a measure on $\mathbb{R}^1$, not as a measure on $\mathbb{R}^N$. In fact, it is easy to see that this cannot be true for the measure on $\mathbb{R}^N$. There is always some $\boldsymbol{h} \in \mathbb{R}^N$ with $\operatorname{Re} h^{(\alpha)} > 0$ for all α and $\zeta = \sum_{\alpha=1}^{N} {h^{(\alpha)}}^2$ real, negative and arbitrary. It follows that zeros of (3.32) could be contained in $\left\{ \boldsymbol{h} \in (\mathbb{C}^n)^N : \operatorname{Re} h^{(\alpha)} > 0 \text{ for all } \alpha \right\}$. This is a consequence of the different symmetry assumption. We therefore must replace the zero-free regions D^n that we encountered in the case $N = 1$ by the regions $L_N^{\pm}$. Their precise form is determined below.

Proof. By rotational invariance of ν, its Laplace transform $\hat{\nu}$ is a function of ζ only. By (3.31), if $h^{(2)} = \cdots = h^{(N)} = 0$, the function $\hat{\nu}$ has only purely imaginary zeros. As in Lemma 3.21, this implies the representation (3.33). In particular, $\hat{\nu}$ is an entire function of ζ only and of order at most 1.

A sequence of polynomials approximating $\hat{\nu}$ can be chosen in the obvious way. The same argument as in Lemma 3.21 together with the following lemma prove $\hat{\nu} \in \overline{\mathscr{P}^N_{1/4\beta+}(L_N)}$. □

Together with the form of $\hat{\nu}$ derived in Proposition 3.22, the next lemma implies that $\hat{\nu}(\boldsymbol{h})$ is nonvanishing whenever $\boldsymbol{h} \in L_N$.

Lemma 3.23. *Let* L_N *be defined by* (3.34) *and let* $\boldsymbol{h} \in L_N$*. Then* $\zeta = \sum_{\alpha=1}^{N} {h^{(\alpha)}}^2$ *is never real and negative.*

Proof. Write $\boldsymbol{h} = (x^{(1)} + iy^{(1)}, \boldsymbol{x} + i\boldsymbol{y})$ where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{N-1}$. Then ζ can be expressed as

$$\zeta = \sum_{\alpha=1}^{N} \left({x^{(\alpha)}}^2 - {y^{(\alpha)}}^2 \right) + i \left(2 \sum_{\alpha=1}^{N} x^{(\alpha)} y^{(\alpha)} \right)$$

with $\operatorname{Re} \zeta = {x^{(1)}}^2 - {y^{(1)}}^2 + |\boldsymbol{x}|^2 - |\boldsymbol{y}|^2$, and $\operatorname{Im} \zeta = 2(x^{(1)} y^{(1)} + \boldsymbol{x} \cdot \boldsymbol{y})$. If $\operatorname{Im} \zeta = 0$, the Cauchy-Schwartz inequality implies $|\boldsymbol{y}| > \left| x^{(1)} \right| \left| y^{(1)} \right| / |\boldsymbol{x}|$, because $0 = \operatorname{Im} \zeta = 2(x^{(1)} y^{(1)} + \boldsymbol{x} \cdot \boldsymbol{y})$ can be rearranged to yield

$$x^{(1)} y^{(1)} = -\boldsymbol{x} \cdot \boldsymbol{y} \leq - |\boldsymbol{x}| \, |\boldsymbol{y}| \, ,$$

where the last inequality is the Cauchy-Schwartz inequality .

The fact that $\boldsymbol{h} \in L_N$ implies $\left| x^{(1)} \right| > |\boldsymbol{y}|$, this is precisely the condition for $\boldsymbol{h} \in L_N$. Inserting into the inequality above and multiplying it by $|\boldsymbol{x}|$ yields $|\boldsymbol{x}| > \left| y^{(1)} \right|$. It follows that

$$\operatorname{Re} \zeta = {x^{(1)}}^2 - |\boldsymbol{y}|^2 > 0. \qquad \square$$

The next steps on the way to a Lee-Yang theorem are only possible for the two-component ferromagnet. We therefore set $N = 2$ from now on and give some remarks about the case $N \geq 3$ in Section 3.6. For $N = 2$, the sets $L_2^{\pm}$ have the simple form

$$L_2^{\pm} = \left\{ \boldsymbol{h}\colon \pm \operatorname{Re} h^{(1)} > \left| \operatorname{Im} h^{(\alpha)} \right| \right\}.$$

Two successive linear transformations of variables allow us to convert the region L_2 into a product of two right half-planes D^n. This equivalence allows us to trace the problem back to the one-component ferromagnet which makes a simple derivation of a Lee-Yang theorem for the case $N = 2$ possible.

For the first transformation, we introduce $\tilde{\boldsymbol{h}} = (h^{(1)}, ih^{(2)})$. Then with

$$\Gamma_{\pm} = \left\{ \boldsymbol{x}\colon \pm x^{(1)} > \left| x^{(2)} \right| \right\},$$

the condition $\boldsymbol{h} \in L_2$ in terms of $\tilde{\boldsymbol{h}}$ becomes Re $\tilde{\boldsymbol{h}} \in \Gamma_+ \cup \Gamma_-$. For the second transformation, define

$$h^{\pm} = 2^{-1/2} \left(\tilde{h}^{(1)} \pm \tilde{h}^{(2)} \right) = 2^{-1/2} \left(h^{(1)} \pm ih^{(2)} \right),$$

so that Re $\tilde{h}_i \in \Gamma_+$ for all i is equivalent to Re $h_i^{\pm} > 0$ for all i. Note that

$$\exp\left(h^{(1)}\phi^{(1)} + h^{(2)}\phi^{(2)} \right) = \exp\left(h^{+}\phi^{-} + h^{-}\phi^{+} \right)$$

where $\phi^{\pm}$ is defined in an analogous way as $h^{\pm}$. For the free partition function it holds that

$$\begin{aligned} \mathscr{Z}_0(h_1^{(1)}, \cdots, h_n^{(2)}) &= \int \exp\left(\sum_{i=1}^{n} \left(h_i^{(1)}\phi_i^{(1)} + h_i^{(2)}\phi_i^{(2)} \right) \right) \prod_{i=1}^{n} \mathrm{d}\nu_i(\boldsymbol{\phi}_i) \\ &= \int \exp\left(\sum_{i=1}^{n} \left(h_i^{+}\phi_i^{-} + h_i^{-}\phi_i^{+} \right) \right) \prod_{i=1}^{n} \mathrm{d}\nu_i(\boldsymbol{\phi}_i). \end{aligned}$$

If we regard the different spin and magnetic field components as separate, independent variables, we see that this is precisely the situation studied before where the summation over n spin sites is replaced by a summation over $2n$ spin sites, the only difference being that differentiation with respect to $h_i^{\pm}$ brings down a factor $\phi_i^{\mp}$. Hence the theorems of Section 3.4 apply.

In terms of the new variables, a ferromagnetic pair interaction is an entire function

$$f(\boldsymbol{\phi}) = \exp\left(\sum_{i,j=1}^{n} \left(J_{ij}^{++}\phi_i^{+}\phi_j^{+} + J_{ij}^{+-}\phi_i^{+}\phi_j^{-} + J_{ij}^{-+}\phi_i^{-}\phi_j^{+} + J_{ij}^{--}\phi_i^{-}\phi_j^{-} \right) \right) \tag{3.35}$$

with all coefficients $J_{ij}^{\pm\pm}, J_{ij}^{\pm\mp} \geq 0$. From our previous considerations, we know that the Lee-Yang theorem in the form of Theorem 3.18 and Corollary 3.19 holds for such an interaction. For a general two-component ferromagnet with partition function given by (3.29), there is therefore also a Lee-Yang theorem, and its precise formulation can be obtained by changing the variables $\phi^{\pm}$ back to $\phi^{(1)}, \phi^{(2)}$.

It is a matter of elementary algebra to express the interaction (3.35) in terms of the variables $\phi_i^{(\alpha)}$,

$$f(\boldsymbol{\phi}) = \exp\Big(\frac{1}{2}\sum_{i,j=1}^{n}\big(\phi_i^{(1)}\phi_j^{(1)}(J_{ij}^{++}+J_{ij}^{+-}+J_{ij}^{-+}+J_{ij}^{--})+\phi_i^{(2)}\phi_j^{(2)}(-J_{ij}^{++}+J_{ij}^{+-}+J_{ij}^{-+}-J_{ij}^{--})$$
$$+i\phi_i^{(1)}\phi_j^{(2)}(J_{ij}^{++}-J_{ij}^{+-}+J_{ij}^{-+}-J_{ij}^{--})+i\phi_i^{(2)}\phi_j^{(1)}(J_{ij}^{++}+J_{ij}^{+-}-J_{ij}^{-+}-J_{ij}^{--})\big)\Big).$$

Comparison with (3.29) yields the conditions

$$J^{(12)} = J_{ij}^{++}-J_{ij}^{+-}+J_{ij}^{-+}-J_{ij}^{--}=0,$$
$$J^{(21)} = J_{ij}^{++}+J_{ij}^{+-}-J_{ij}^{-+}-J_{ij}^{--}=0,$$

from which $J^{++}=J^{--}$ and $J^{+-}=J^{-+}$ follows. The expressions for $J^{(11)}$ and $J^{(22)}$ simplify,

$$J^{(11)} = J^{++}+J^{+-}, \quad \text{and} \quad J^{(22)} = J^{+-}-J^{++}.$$

Recall that a Lee-Yang theorem holds for the interaction (3.35) whenever $J^{++}, J^{+-} \geq 0$ (assuming that $J^{++}=J^{--}$ and $J^{+-}=J^{-+}$). We deduce that for the interaction

$$f(\boldsymbol{\phi}) = \exp\left(\sum_{i=1}^{n}\sum_{\alpha=1}^{2} J_{ij}^{(\alpha\alpha)}\phi_i^{(\alpha)}\phi_j^{(\alpha)}\right), \tag{3.36}$$

a Lee-Yang theorem holds if

$$J_{ij}^{(11)} \geq \left|J_{ij}^{(22)}\right| \quad \text{for all } i,j. \tag{3.37}$$

The following analogues of Theorem 3.18 and Corollary and 3.19 now immediately follow.

Theorem 3.24. *Let $\mu_0 \not\equiv$ be a finite (positive) measure on $\mathbb{R}^{2n}$, $\mu_0 \in \mathscr{T}_\beta^{2n}$ and $\hat{\mu}_0 \in \overline{\mathscr{P}_{1/4\beta+}^{2n}((L_2^+)^n)}$. Now let $f \in \overline{\mathscr{P}_{\alpha+}^{2n}((L_2^+)^n)}$ with $\alpha < \beta$ be nonnegative on the support of μ_0 and strictly positive on a set of nonzero μ_0-measure.*

Then $\mu = f\mu_0$ is a finite positive measure with $\mu \not\equiv 0, \mu \in \mathscr{T}_\gamma^{2n}$ and $\hat{\mu} \in \overline{\mathscr{P}_{1/4\gamma+}^{2n}((L_2^+)^n)}$ for every $\gamma < \beta - \alpha$ (and $\gamma = \infty$ if $\beta = \infty$). In particular, we can take f of the form (3.36)*, provided $\alpha = \|J\| < \beta$.*

Corollary 3.25. *For $1 \leq i \leq n$, let $\nu_i \in \mathscr{T}_\beta^2$ be a rotationally invariant measure on $\mathbb{R}^2$ satisfying $\int e^{h\phi^{(1)}}\mathrm{d}\nu(\boldsymbol{\phi}) \neq 0$ for* Re $h \neq 0$*. Let J be a symmetric $2n\times 2n$ matrix satisfying* (3.37) *and $\|J\| < \beta$.*

Then for the measure μ on $\mathbb{R}^{2n}$ given by

$$\mathrm{d}\mu(\boldsymbol{\phi}) = \exp\left(\sum_{i,j=1}^{n}\sum_{\alpha=1}^{2} J_{ij}^{(\alpha\alpha)}\phi_i^{(\alpha)}\phi_j^{(\alpha)}\right)\prod_{i=1}^{n}\mathrm{d}\nu_i(\boldsymbol{\phi}_i)$$

it holds that $\mu \not\equiv 0, \mu \in \mathscr{T}_\gamma^{2n}$ and $\hat{\mu} \in \overline{\mathscr{P}_{1/4\gamma+}^{2n}((L_2^+)^n)}$ for every $\gamma < \beta - \|J\|$ (and $\gamma = \infty$ if $\beta = \infty$).

In particular, the partition function

$$\hat{\mu}(\boldsymbol{h}) = \int \exp\left(\sum_{i=1}^{n}\sum_{\alpha=1}^{2} h_i^{(\alpha)}\phi_i^{(\alpha)}\right) \mathrm{d}\mu(\boldsymbol{\phi})$$

is nonvanishing whenever

$$\mathrm{Re}\ h_i^{(1)} > \left|\mathrm{Im}\ h_i^{(2)}\right| \quad \textit{for all } i.$$

3.6 A note on ferromagnets with more than two spin components

Unlike for the two-component model, extending the result to a Lee-Yang theorem for general N-component ferromagnets for $N \geq 3$ with a partition function analogous to (3.29) is not as straightforward. Although a thorough analysis of the $N \geq 3$ case is beyond the scope of this thesis, the findings presented in Section 3.5 provide some insight into the challenges that arise here.

Proposition 3.22 hold for all $N \geq 2$. In particular, the zero-free region always contains the sets

$$L_N^{\pm} = \left\{\boldsymbol{h}\colon\ \pm\,\mathrm{Re}\ h^{(1)} > \left[\sum_{\alpha=2}^{N}(\mathrm{Im}\ h^{(\alpha)})^2\right]^{1/2}\right\}.$$

In the case $N = 2$, the sum consists of a single term and the sets $L_2^{\pm}$ therefore assume the form

$$L_2^{\pm} = \left\{\boldsymbol{h}\colon\ \pm\,\mathrm{Re}\ h^{(1)} > \left|\mathrm{Im}\ h^{(\alpha)}\right|\right\}.$$

In the preceding paragraph, it was shown that this form allows for a linear transformation of these sets into a product of two sets D^n for which the Lee-Yang theorem has been proven in Section 3.4. If we take $N \geq 3$, the condition for $\mathbf{h} = (h^{(1)}, \cdots, h^{(N)})$ to be in $L_N^{\pm}$ is no longer linear which makes such a transformation impossible. As a result, Propositions 3.1 and 3.11 do not hold in an equally obvious way and have to be replaced by analogous statements involving the sets $L_N^{\pm}$. Lieb and Sokal also provided a proof for adapted versions of these two propositions, showing that for an N-component ferromagnet whose interaction coefficients satisfy

$$J_{ij}^{(11)} \geq \sum_{\alpha=2}^{N}\left|J_{ij}^{(\alpha\alpha)}\right| \quad \text{for all } i, j, \tag{3.38}$$

a Lee-Yang theorem analogous to Corollaries 3.19 and 3.25 holds. However, the condition (3.38) is generally regarded unsatisfactory [LS81; FR12]. One would like to replace it by the condition

$$J_{ij}^{(11)} \geq \max_{2\leq\alpha\leq N}\left|J_{ij}^{(\alpha\alpha)}\right| \quad \text{for all } i, j, \tag{3.39}$$

which is known to imply a Lee-Yang theorem for certain multi-component ferromagnetic models such as the Heisenberg model [Asa70b; SF71; Dun79]. The question of whether the condition (3.39) is generally sufficient to imply a Lee-Yang theorem remains open, leaving room for future research.

Appendix

A Facts about Fréchet spaces

We review the definition of Fréchet spaces and summarise the relevant properties for our proof. This short review follows [EW17], pp. 295.

Definition A.1. A *Fréchet space* is a locally convex vector space X whose topology is induced by countably many seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$, meaning that a subset U is open if and only if for all $u \in U$ there are constants $K \geq 0, r > 0$ such that $\{v\colon \|v-u\|_k < r \text{ for all } k \geq K\}$ is a subset of U, and X is complete.

The topology of a Fréchet space X is equivalently induced by the metric ([EW17] 8.65)

$$d(x,y) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{\|x-y\|_n}{1+\|x-y\|_n}.$$

Note that d does not characterise bounded sets if the space X is metrisable but not normable (for otherwise, there would be a bounded convex neighbourhood of the origin, which, by Kolmogorov's normability criterion (for example p. 484 in [Wer18]) would imply that the space is normable). Instead boundedness is characterised by the seminorms (see pp. 49 in [Sim17]). Our Fréchet spaces $\mathscr{A}^n_{a+}$ are not normable.

Definition A.2. Let X be a Fréchet space whose topology is induced by the seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$. A sequence $(x_j)_{j\in\mathbb{N}}$ is *bounded* if $\sup_{j\in\mathbb{N}} \|x_j\|_n < \infty$ for all $n \in \mathbb{N}$.

The next proposition characterises convergence in Fréchet spaces.

Proposition A.3. *A sequence $(f_j)_{j\in\mathbb{N}}$ in X converges to $f \in X$ if and only if $\|f_j - f\|_n \to 0$ as $j \to \infty$ for all n.*

The next proposition follows from the fact that a Fréchet space is metrisable (see Theorem 2.10 in [Sim17]).

Proposition A.4. *Let X be a Fréchet space and $A \subset X$ closed. Then A is sequentially closed.*

B Proof of the identity (3.18)

Lemma B.1. *Let g be an entire function and $t \in \mathbb{C}$. Then for all $z \in \mathbb{C}$ it holds that*

$$\exp(2t\partial_z)g(z) = g(z+2t).$$

Proof. The identity can be proven by writing out the differential operator as the power series by which it is defined, and then applying it termwise to the power series of f. Since f is an entire function, its power series is absolutely convergent everywhere, this is important for the proof.

Writing both sums out yields

$$\begin{aligned}\exp(2t\partial_z)f(z) &= \sum_{n\in\mathbb{N}_0}\frac{(2t\partial_z)^n}{n!}f(z)\\ &= \sum_{n\in\mathbb{N}_0}\frac{(2t\partial_z)^n}{n!}\sum_{k\in\mathbb{N}_0}\frac{f^{(k)}(z_0)(z-z_0)^k}{k!}.\end{aligned}$$

Now we determine the result of an application of a single derivative $2t\partial_z$ and from this the application of arbitrary powers of it to the power series of f. We obtain

$$\begin{aligned}(2t\partial_z)\sum_{k\in\mathbb{N}_0}\frac{f^{(k)}(z_0)(z-z_0)^k}{k!} &= \sum_{k\in\mathbb{N}}\frac{f^{(k)}(z_0)(z-z_0)^{k-1}}{(k-1)!},\\ (2t\partial_z)^n\sum_{k\in\mathbb{N}_0}2t\frac{f^{(k)}(z_0)(z-z_0)^k}{k!} &= \sum_{k\in\mathbb{N}_{\geq n}}\frac{(2t)^n f^{(k)}(z_0)(z-z_0)^{k-n}}{(k-n)!}.\end{aligned}$$

To prove the lemma, the only remaining step is to carry out the summation over n,

$$\begin{aligned}\sum_{n\in\mathbb{N}_0}\frac{(2t\partial_z)^n}{n!}\sum_{k\in\mathbb{N}_0}\frac{f^{(k)}(z_0)(z-z_0)^k}{k!} &= \sum_{k\in\mathbb{N}_0}\sum_{n=0}^{k}\frac{(2t)^n f^{(k)}(z_0)(z-z_0)^{k-n}}{n!(k-n)!}\\ &= \sum_{k\in\mathbb{N}}\frac{f^{(k)}(z_0)}{k!}\sum_{n=0}^{k}\frac{k!}{n!(k-n)!}(2t)^n(z-z_0)^{k-n}\\ &= \sum_{k\in\mathbb{N}}\frac{f^{(k)}(z_0)}{k!}(2t+z-z_0)^k = f(2t+z).\end{aligned}$$

□

C Proof of Lemma 3.14

We first prove the lemma and afterwards give some lemmas that we use in the proof of Lemma 3.14.

Lemma C.1. *Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = N$, $M \in \mathbb{N}_0$, $z \in \mathbb{C}^n$ and $f_z(x) = \exp\left(zx - ax^2\right)$. Then there is a constant C (dependent on M, N) such that the following inequality holds*

$$\sup_{x\in\mathbb{R}^n} \left(1 + |x|^M\right) |\partial_x^\alpha f_z(x)| \leq C \left(1 + |z|^{N+M+1}\right) \exp\left(\frac{|z|^2}{4a}\right).$$

Proof. Note that for $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and its Fourier transform $\mathcal{F}(\varphi)$, there exists a constant C_0 such that the following holds (see Theorem 1.4 in [Kat68]),

$$\|\varphi\|_\infty \leq C_0 \|\mathcal{F}(\varphi)\|_1 .$$

Moreover,

$$\partial_x^\alpha \varphi(x) = \int_{\mathbb{R}^n} (ip)^\alpha e^{ipx} \mathcal{F}(\varphi)(p) \frac{\mathrm{d}^n p}{(2\pi)^n},$$

so that $(\mathcal{F}(\partial_x^\alpha \varphi))(p) = (ip)^\alpha \mathcal{F}(\varphi)(p)$, and it follows

$$\|\partial^\alpha \varphi\|_\infty \leq C_0 \|(ip)^\alpha \mathcal{F}(\varphi)\|_1 .$$

Analogously, letting $\psi_\alpha(x) = x^\alpha$, then

$$\partial_p^\alpha \mathcal{F}(\varphi)(p) = \int_{\mathbb{R}^n} (-ix)^\alpha \varphi(x) e^{-ipx} \mathcal{F}(\varphi)(p) \mathrm{d}^n x = (-i)^\alpha (\mathcal{F}(\psi_\alpha \varphi))(p).$$

It follows that

$$\|x^\alpha \varphi\|_\infty = \|\psi_\alpha \varphi\|_\infty \leq C_0 \left\|\partial_p^\alpha \mathcal{F}(\varphi)\right\|_1 .$$

The expression$\left(1 + |x|^M\right) |\partial_x^\alpha f_z(x)|$ can thus be bounded using the Fourier transform of f_z,

$$\mathcal{F}(f_z)(p) = \left(\frac{\pi}{a}\right)^{n/2} e^{-\frac{1}{4a}(p-iz)^2}.$$

As we aim to bound the integral of $|\mathcal{F}(f_z)|$, we need the identity

$$\left| e^{-\frac{1}{4a}(p-iz)^2} \right| = e^{-\frac{1}{4a}(p+\mathrm{Im}z)^2} e^{\frac{(\mathrm{Re}z)^2}{4a}}.$$

Moreover, the exponential can be factorised,

$$e^{-\frac{1}{4a}(p+\mathrm{Im}z)^2} = \prod_{\mu=1}^{n} e^{-\frac{1}{4a}(p_\mu + \mathrm{Im}z_\mu)^2}.$$

We then calculate, using the substitution $\xi = p + \text{Im } z$,

$$
\begin{aligned}
\|\partial_x^\alpha f_z\|_\infty &\leq C_0 \left(\frac{\pi}{a}\right)^{n/2} \int_{\mathbb{R}^n} |p^\alpha| \left| e^{-\frac{1}{4a}(p-iz)^2} \right| \mathrm{d}^n p \\
&= C_0 \left(\frac{\pi}{a}\right)^{n/2} e^{\frac{(\mathrm{Re} z)^2}{4a}} \prod_{\mu=1}^{n} \int_{\mathbb{R}} |p_\mu|^{\alpha_\mu} e^{-\frac{1}{4a}(p_\mu + \mathrm{Im} z_\mu)^2} \mathrm{d}p_\mu \\
&= C_0 \left(\frac{\pi}{a}\right)^{n/2} e^{\frac{(\mathrm{Re} z)^2}{4a}} \prod_{\mu=1}^{n} \int_{\mathbb{R}} |\xi_\mu - \text{Im } z|^{\alpha_\mu} e^{-\frac{1}{4a}(\xi_\mu)^2} \mathrm{d}\xi_\mu .
\end{aligned}
$$

With Lemma C.5, we can bound the integrals, which yields

$$
\|\partial_x^\alpha f_z\|_\infty \leq C_0 \left(\frac{\pi}{a}\right)^{n/2} e^{\frac{(\mathrm{Re} z)^2}{4a}} \prod_{\mu=1}^{n} K_\mu \left(1 + |\text{Im } z_\mu|\right)^{\alpha_\mu}
$$

where K_μ are the respective constants obtained from the application of Lemma C.5 for the integral over ξ_μ. Furthermore, with $|\alpha| = \sum_{\mu=1}^{n} \alpha_\mu$, it holds that

$$
|p^\alpha| = \prod_{\mu=1}^{n} |p_\mu|^{\alpha_\mu} \leq \prod_{\mu=1}^{n} |p|^{\alpha_\mu} = |p|^{|\alpha|} , \tag{C.1}
$$

Using (C.1), the fact that $|\text{Im } z|, \ |\text{Re } z| \leq |z|$ and Lemma C.4, we find

$$
\|\partial_x^\alpha f_z\|_\infty \leq C_1 e^{\frac{(|z|)^2}{4a}} \left(1 + |z|^{|\alpha|}\right),
$$

where C_1 is some suitably chosen constant.

Similarly, assuming that M is even and choosing $\beta \in \mathbb{N}_0^n$ so that $|x|^M = x^\beta$, we see that

$$
\begin{aligned}
\left\| |x|^M \partial_x^\alpha f_z \right\|_\infty &\leq C_0 \left(\frac{\pi}{a}\right)^{n/2} \int_{\mathbb{R}^n} \left| \partial_p^\beta \left((ip)^\alpha e^{-\frac{1}{4a}(p-iz)^2} \right) \right| \mathrm{d}^n p \\
&= C_0 \left(\frac{\pi}{a}\right)^{n/2} \int_{\mathbb{R}^n} \left| P_{\alpha+\beta}(p,z) e^{-\frac{1}{4a}(p-iz)^2} \right| \mathrm{d}^n p
\end{aligned}
$$

where $P_{\alpha+\beta}(p, z)$ is a polynomial of the form

$$
P_{\alpha+\beta}(p,z) = \sum_{\substack{\boldsymbol{k} \in N_0^n, \\ |\boldsymbol{k}| \leq \deg P}} c_{\boldsymbol{k}} p^{\gamma_{\boldsymbol{k}}^{(p)}} z^{\gamma_{\boldsymbol{k}}^{(z)}}
$$

and $\gamma_{\boldsymbol{k}}^{(p)} + \gamma_{\boldsymbol{k}}^{(z)} \leq \alpha + \beta$ entry-wise. We therefore write

$$
\left\| |x|^M \partial_x^\alpha f_z \right\|_\infty \leq C_0 \left(\frac{\pi}{a}\right)^{n/2} \sum_{\substack{\boldsymbol{k} \in N_0^n, \\ |\boldsymbol{k}| \leq \deg P}} |c_{\boldsymbol{k}}| \int_{\mathbb{R}^n} \left| p^{\gamma_{\boldsymbol{k}}^{(p)}} z^{\gamma_{\boldsymbol{k}}^{(z)}} e^{-\frac{1}{4a}(p-iz)^2} \right| \mathrm{d}^n p
$$

and then use the same arguments as before to estimate for each $\boldsymbol{k}$ separately

$$\int_{\mathbb{R}^n} \left| p^{\gamma_{\boldsymbol{k}}^{(p)}} z^{\gamma_{\boldsymbol{k}}^{(z)}} e^{-\frac{1}{4a}(p-iz)^2} \right| \mathrm{d}^n p \leq \left| z^{\gamma_{\boldsymbol{k}}^{(z)}} \right| e^{\frac{(\mathrm{Re} z)^2}{4a}} \prod_{\mu=1}^{n} K'_\mu \left(1 + |\mathrm{Im}\ z_\mu|\right)^{\gamma_{\boldsymbol{k}\mu}^{(p)}} .$$

Obviously, $|\mathrm{Im}\ z|, |\mathrm{Re}\ z| \leq |z|$, and it holds that

$$|p^\alpha| = \prod_{\mu=1}^{n} |p_\mu|^{\alpha_\mu} \leq \prod_{\mu=1}^{n} |p|^{\alpha_\mu} = |p|^{|\alpha|}$$

with $|\alpha| = \sum_{\mu=1}^{n} \alpha_\mu$, so we can further estimate, employing (C.1) and Lemma C.4,

$$\left| z^{\gamma_{\boldsymbol{k}}^{(z)}} \right| e^{\frac{(\mathrm{Re} z)^2}{4a}} \prod_{\mu=1}^{n} K'_\mu \left(1 + |\mathrm{Im}\ z_\mu|\right)^{\gamma_{\boldsymbol{k}\mu}^{(p)}} \leq C_2 (e^{\frac{|z|^2}{z}})^2 4a\, |z|^{\left|\gamma_{\boldsymbol{k}}^{(z)}\right|} \left(1 + |z|^{\left|\gamma_{\boldsymbol{k}}^{(p)}\right|}\right) .$$

We use this, Lemma C.4 and $\left|\gamma_{\boldsymbol{k}}^{(p)}\right| + \left|\gamma_{\boldsymbol{k}}^{(z)}\right| \leq |\alpha| + |\beta|$ to get an overall estimate

$$\left\| |x|^M \partial_x^\alpha f_z \right\|_\infty \leq C_3 e^{\frac{|z|^2}{4a}} \left(1 + |z|^{|\alpha|+|\beta|}\right) .$$

Now we put the separate estimates together to show that

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} \left(1 + |x|^N\right) \partial_x^\alpha f_z(x) &\leq \|\partial_x^\alpha f_z\|_\infty + \left\| |x|^N \partial_x^\alpha f_z \right\|_\infty \\
&\leq \left(C_1 e^{\frac{|z|^2}{4a}} \left(1 + |z|^{|\alpha|}\right) + C_3 e^{\frac{|z|^2}{4a}} \left(1 + |z|^{|\alpha|+|\beta|}\right) \right) \\
&\leq C_4 \left(1 + |z|^{|\alpha|+|\beta|}\right) e^{\frac{|z|^2}{4a}} ,
\end{aligned}$$

where Lemma C.4 has been used in the last step.

Moreover, Lemma C.3 shows that there exists a constant C_5 such that

$$\left(1 + |z|^{|\alpha|+|\beta|}\right) \leq C_5 \left(1 + |z|\right)^{|\alpha|+|\beta|} .$$

All $z \in \mathbb{C}^n$ satisfy $1 + |z| \geq 1$. Since $y \mapsto x^y$ is monotonously increasing for $x \in [1, \infty)$ and $y \geq 0$, it is clear that

$$(1 + |z|)^{|\alpha|+|\beta|} \leq (1 + |z|)^{|\alpha|+|\beta|+1} .$$

Finally, Lemma C.3 yields a constant $C_6 > 0$ such that

$$(1 + |z|)^{|\alpha|+|\beta|+1} \leq C_6^{-1} \left(1 + |z|^{|\alpha|+|\beta|+1}\right) .$$

Recalling $|\alpha| = N, |\beta| = M$ finishes the proof in the case that M is even.

If M is uneven, we replace M by $M + 1$ and repeat the above argument but leave out the step where we establish that $\left(1+|z|^{|\alpha|+|\beta|}\right) \leq \frac{C_5}{C_6}\left(1+|z|^{|\alpha|+|\beta|+1}\right)$. This completes the proof for uneven M. □

Now we prove somme lemmas that have been used in the above proof.

Lemma C.2. *Let* $p = \sum_{i=0}^{n} a_i x^i$ *be a polynomial in a real variable* x*, where* $n := \deg p$*. Assume* $a_i \geq 0$ *for all* $i = 0, \cdots, n$*. Then there are constants* $0 \leq c < C$ *such that* $c(1+x)^n \leq p(x) \leq C(1+x)^n$ *for all* $x \geq 0$*.*

Proof. Obviously, $p(x) \geq 0$ whenever $x \geq 0$, so it can be taken $c = 0$. In order to find a suitable constant for the upper bound, write $(1+x)^n = \sum_{i=0}^{n} b_i x^i$. Note that all coefficients b_i are nonzero, so that $c_i := \frac{a_i}{b_i}$ is well-defined. Let $C := \max_{1 \leq i \leq n} c_i$, then $C(1+x)^n = C\sum_{i=0}^{n} b_i x^i \geq \sum_{i=0}^{n} a_i x^i = p(x)$, this yields the desired inequality. □

Lemma C.3. *Let* $M \in \mathbb{N}_0$*. There are constants* $0 < c \leq C$ *(that are dependent on* M*) such that for all* $x \in \mathbb{R}$ *the following holds*

$$c\,(1+|x|)^M \leq \left(1+|x|^M\right) \leq C\,(1+|x|)^M .$$

Proof. If $M = 0$, we can take $c = C = 2$. If $M = 1$, we can take $c = C = 1$. In both cases, that finishes the proof.

Let therefore $M \geq 2$ and define

$$g : (-1,\infty) \to \mathbb{R}, x \mapsto \frac{(1+x)^M}{1+x^M}.$$

Then g is a continuously differentiable function and $g(0) = 1$, $\lim_{x\to\infty} g(x) = 1$. Now consider

$$\frac{\mathrm{d}g}{\mathrm{d}x}(x) = \frac{M(1+x)^{M-1}\left(1-x^{M-1}\right)}{\left(1+x^M\right)^2}.$$

It is easy to see that $g'(x) = 0$ holds if and only if $x = 1$ and that therefore, $1 \leq g(x) \leq g(1) = 2^{M-1}$ for all $x \in [0,\infty)$. As $g(0) < g(1)$ and $\lim_{x\to\infty} g(x) < g(1)$, it is evident that there is a maximum at $x = 1$. This yields the desired inequality with $c = 2^{1-M}, C = 1$. □

Lemma C.4. *For any polynomial* p *in* n *variables* $x_1, \cdots, x_n$*,* $\sum_{\boldsymbol{k}=0}^{\deg P} \prod_{i=1}^{n} c_{k_i} x_i^{k_i}$*, there are constants* $0 \leq c < C$ *such that*

$$c\left(1+\left|x^k\right|\right) \leq p(|x_1|, \cdots, |x_1|) \leq C\left(1+\left|x^k\right|\right).$$

Proof. This is an immediate consequence of Lemma C.2 and Lemma C.3. □

Lemma C.5. *Let $a \in \mathbb{R}, a > 0$. For all $b \in \mathbb{R}, m \in \mathbb{N}_0$, there exists a constant K (that depends on a, b and m) such that*

$$0 \leq \int_{\mathbb{R}} |p|^m e^{-\frac{1}{4a}(p+b)^2} \mathrm{d}p \leq K (1 + |b|)^m .$$

Proof. Since the integrand is always nonnegative, it is clear that

$$0 \leq \int_{\mathbb{R}} |p|^m e^{-\frac{1}{4a}(p+b)^2} \mathrm{d}p.$$

For the other inequality, use the substitution $\xi = p + b$ to write

$$\begin{aligned}
\int_{\mathbb{R}} |p|^m e^{-\frac{1}{4a}(p+b)^2} \mathrm{d}p &= \int_{\mathbb{R}} |\xi - b|^m e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi \\
&= \int_{-\infty}^{b} (b-\xi)^m e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi + \int_{b}^{\infty} (\xi - b)^m e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi \\
&= \sum_{k=0}^{m} \binom{m}{k} \left[b^{m-k} \int_{-b}^{\infty} \xi^k e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi + (-b)^{m-k} \int_{b}^{\infty} \xi^k e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi \right].
\end{aligned}$$

We therefore need to evaluate the following integral for $c = \pm b$,

$$\int_{c}^{\infty} \xi^k e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi = \begin{cases} (-4a)^{k/2} \partial_{1/4a}^{k/2} \int_{c}^{\infty} e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi & \text{if } k \text{ is even.} \\ (-4a)^{(k-1)/2} \partial_{1/4a}^{(k-1)/2} \int_{c}^{\infty} \xi e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi & \text{if } k \text{ is uneven.} \end{cases}$$

It is clear that

$$\int_{c}^{\infty} e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi \leq \int_{-\infty}^{\infty} e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi = \sqrt{4a\pi}$$

and

$$\int_{c}^{\infty} \xi e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi = \left[-2ae^{-\frac{1}{4a}\xi^2} \right]_{c}^{\infty} = 2ae^{-\frac{c^2}{4a}} \leq 2a.$$

This shows that there are constants $K_{k,c}^a$ for $k \in \mathbb{N}_0, c \in \mathbb{R}$ such that

$$\int_{c}^{\infty} \xi^k e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi \leq K_{k,c}^a.$$

Inserting into the integral that we aim to bound yields

$$\begin{aligned}
\int_{\mathbb{R}} |p|^m \, e^{-\frac{1}{4a}(p+b)^2} \mathrm{d}p &= \sum_{k=0}^{m} \binom{m}{k} \left[b^{m-k} \int_{-b}^{\infty} \xi^k e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi + (-b)^{m-k} \int_{b}^{\infty} \xi^k e^{-\frac{1}{4a}\xi^2} \mathrm{d}\xi \right] \\
&= \sum_{k=0}^{m} \binom{m}{k} \left[b^{m-k} K_{k,-b} + (-b)^{m-k} K_{k,b} \right] \\
&\leq \sum_{k=0}^{m} \binom{m}{k} \left[|b|^{m-k} \, |K_{k,-b}| + |b|^{m-k} \, |K_{k,b}| \right] \\
&\leq \max_{k \leq m} \{ |K_{k,b}| \, , |K_{k,-b}| \} \sum_{k=0}^{m} \binom{m}{k} 1^k \, |b|^{m-k} = K \, (1 + |b|)^m \, ,
\end{aligned}$$

where $K = \max_{k \leq m} \{ K^a_{k,b}, K^a_{k,-b} \}$. □

References

[Asa70a] Asano, T. “Lee-Yang Theorem and the Griffiths Inequality for the Anisotropic Heisenberg Ferromagnet”. In: *Physical Review Letters* **24** (1970), pp. 1409–1411.

[Asa70b] Asano, T. “Theorems on the Partition Functions of the Heisenberg Ferromagnets”. In: *Journal of the Physical Society of Japan* **29** (1970), pp. 350–359.

[BDL05] Bena, I., Droz, M., and Lipowski, A. “Statistical Mechanics of Equilibrium and Nonequilibrium Phase Transitions: The Yang-Lee Formalism”. In: *International Journal of Modern Physics B* **19** (2005), pp. 4269–4329.

[BK90] Borgs, C. and Koteckỳ, R. “A Rigorous Theory of Finite-Size Scaling at First-Order Phase Transitions”. In: *Journal of Statistical Physics* **61** (1990), pp. 79–119.

[Bri22] Bricmont, J. *Making Sense of Statistical Mechanics.* (1. ed.). Cham: Springer International Publishing, 2022.

[Cal85] Callen, H. B. *Thermodynamics and an Introduction to Thermostatistics.* (2. ed.). New York: Wiley, 1985.

[CMÖ21] Chirilă, A., Marin, M., and Öchsner, A. *Distribution Theory Applied to Differential Equations.* Cham: Springer, 2021.

[DDH02] Dammer, S. M., Dahmen, S. R., and Hinrichsen, H. “Yang-Lee Zeros for a Nonequilibrium Phase Transition”. In: *Journal of Physics A: Mathematical and General* **35** (2002), pp. 4527–4539.

[Dav02] Davies, B. *Integral Transforms and their Applications.* (3. ed.). New York: Springer, 2002.

[Dob68] Dobrushin, R. L. “The Problem of Uniqueness of a Gibbsian Random Field and the Problem of Phase Transitions”. In: *Functional Analysis and its Applications* **2** (1968), 302—312.

[Dun79] Dunlop, F. “Analyticity of the Pressure for Heisenberg and Plane Rotator Models”. In: *Communications in Mathematical Physics* **69** (1979), pp. 81–88.

[EW17] Einsiedler, M. and Ward, T. *Functional Analysis, Spectral Theory, and Applications.* Cham: Springer, 2017.

[Fre14] Freitag, E. *Funktionentheorie 2. Riemann’sche Flächen Mehrere Komplexe Variable Abel’sche Funktionen Höhere Modulformen.* (2., überarb. Aufl.). Berlin: Springer, 2014.

[FB06] Freitag, E. and Busam, R. *Funktionentheorie 1.* (4., korr. und erw. Aufl.). Berlin: Springer, 2006.

[FV18] Friedli, S. and Velenik, Y. *Statistical Mechanics of Lattice Systems. A Concrete Mathematical Introduction.* Cambridge: Cambridge University Press, 2018.

[FR12] Fröhlich, J. and Rodriguez, P.-F. “Some Applications of the Lee-Yang Theorem”. In: *Journal of Mathematical Physics* **53** (2012), p. 095218.

[Gal99] Gallavotti, G. *Statistical Mechanics. A Short Treatise.* Berlin: Springer, 1999.

[GS64] Gelfand, I. M. and Shilov, G. E. *Generalized Functions. Volume I: Properties and Operations.* Translated from Russian. New York: Academic Press, 1964.

[Gri64] Griffiths, R. B. "Peierls Proof of Spontaneous Magnetization in a Two-Dimensional Ising Ferromagnet". In: *Physical Review* **136** (1964), A437–A439.

[GRS75] Guerra, F., Rosen, L., and Simon, B. "The $P(\varphi)_2$ Euclidean Quantum Field Theory as Classical Statistical Mechanics". In: *Annals of Mathematics* (1975), pp. 111–189.

[Hua63] Huang, K. *Statistical Mechanics.* New York: Wiley, 1963.

[Isi25] Ising, E. "Beitrag zur Theorie des Ferromagnetismus". In: *Zeitschrift für Physik* **31** (1925), pp. 253–258.

[Isi+17] Ising, T., Folk, R., Kenna, R., Berche, B., and Holovatch, Y. "The Fate of Ernst Ising and the Fate of his Model". In: *Journal of Physical Studies* **21** (2017), p. 3002.

[Kat68] Katznelson, Y. *An Introduction to Harmonic Analysis.* New York: Wiley, 1968.

[LB99] Lavis, D. A. and Bell, G. M. *Closed Form and Exact Solutions.* (2., enl. and rev. ed.). Berlin: Springer, 1999.

[LY52a] Lee, T. D. and Yang, C. N. "Statistical Theory of Equations of State and Phase Transitions. I. Theory of Condensation". In: *Physical Review* **87** (1952), pp. 404–409.

[LY52b] Lee, T. D. and Yang, C. N. "Statistical Theory of Equations of State and Phase Transitions. II. Lattice Gas and Ising Model". In: *Physical Review* **87** (1952), pp. 410–419.

[Len73] Lenard, A., ed. *Statistical Mechanics and Mathematical Problems.* Berlin: Springer, 1973.

[Lev64] Levin, B. J. *Distribution of Zeros of Entire Functions.* Translated from the Russian. Providence, RI: American Mathematical Society, 1964.

[LS81] Lieb, E. H. and Sokal, A. D. "A General Lee-Yang Theorem for One-Component and Multicomponent Ferromagnets". In: *Communications in Mathematical Physics* **80** (1981), pp. 153–179.

[Lip22] Lipowski, A. "Ising Model: Recent Developments and Exotic Applications". In: *Entropy* **24** (2022).

[MW19] Ma, T. and Wang, S. *Phase Transition Dynamics.* (2. ed.). Cham: Springer, 2019.

[Mar66] Marden, M. *Geometry of Polynomials.* (2. ed.). Providence, RI: American Mathematical Society, 1966.

[Min00] Minlos, R. A. *Introduction to Mathematical Statistical Physics.* Providence, RI: American Mathematical Society, 2000.

[Mon91] Monroe, J. L. "Restrictions on the Phase Diagrams for a Large Class of Multisite Interaction Spin Systems." In: *Journal of Statistical Physics* **65** (1991), 445—452.

[MH77] Müller-Hartmann, E. "Theory of the Ising Model on a Cayley Tree". In: *Zeitschrift für Physik B Condensed Matter and Quanta* **27** (1977), pp. 161–168.

[New74] Newman, C. M. "Zeros of the Partition Function for Generalized Ising Systems". In: *Communications on Pure and Applied Mathematics* **27** (1974), pp. 143–159.

[Oli17] Oliveira, M. J. de. *Equilibrium Thermodynamics.* (2. ed.). Berlin: Springer, 2017.

[Ons44] Onsager, L. "Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition". In: *Physical Review* **65** (1944), pp. 117–149.

[Pei36] Peierls, R. E. "On Ising's Model of Ferromagnetism". In: *Mathematical Proceedings of the Cambridge Philosophical Society* **32** (1936), pp. 477–481.

[RS72] Reed, M. and Simon, B. *Functional Analysis.* New York: Academic Press, 1972.

[Rob55] Robbins, H. "A Remark on Stirling's Formula". In: *The American Mathematical Monthly* **62** (1955), pp. 26–29.

[Rue71] Ruelle, D. "Extension of the Lee-Yang Circle Theorem". In: *Physical Review Letters* **26** (1971), pp. 303–304.

[Rue73] Ruelle, D. "Some Remarks on the Location of Zeroes of the Partition Function for Lattice Systems". In: *Communications in Mathematical Physics* **31** (1973), 265——277.

[Rue07] Ruelle, D. *Statistical Mechanics. Rigorous Results.* (Reprint). Singapore: World Scientific, 2007.

[Sim17] Simon, J. C. H., ed. *Banach, Fréchet, Hilbert and Neumann spaces.* London: ISTE Ltd, 2017.

[Sta08] Stauffer, D. "Social Applications of Two-Dimensional Ising Models". In: *American Journal of Physics* **76** (2008), pp. 470–473.

[Str03] Strichartz, R. S. *A Guide to Distribution Theory and Fourier Transforms.* River Edge, NJ: World Scientific Publishing, 2003.

[Suz68] Suzuki, M. "Theorems on Extended Ising Model with Applications to Dilute Ferromagnetism". In: *Progress of Theoretical Physics* **40** (1968), pp. 1246–1256.

[SF71] Suzuki, M. and Fisher, M. E. "Zeros of the Partition Function for the Heisenberg, Ferroelectric, and General Ising Models". In: *Journal of Mathematical Physics* **12** (1971), pp. 235–246.

[Wer18] Werner, D. *Funktionalanalysis.* (8. Aufl.). Berlin: Springer, 2018.

[ZJ21] Zinn-Justin, J. *Quantum Field Theory and Critical Phenomena.* (5. ed.). Oxford: Oxford University Press, 2021.